Thompson's Group F

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Introduction

Definitions:

- A **homeomorphism** is a continuous bijection whose inverse is also continuous.
- An **orientation-preserving** hoomeomorphism of $[0,1]$ to $[0,1]$ implies 0 maps to 0 and 1 maps to 1.
- A **piecewise-linear** homeomorphism can be thought of as a piecewise function whose pieces are lines.
- The **dyadic rationals** are rational numbers of the form $\frac{p}{q}$ where $q = 2^k$ for some $k \in \mathbb{Z}$.

Thompson's Group F

Under the operation of function composition, the piecewise-linear, orientation preserving homeomorphisms from $[0,1]$ to itself whose breakpoints occur at dyadic rationals form a *group*.

An Example:

$$x_1(t) = \begin{cases} t & \text{for } 0 \le t \le \frac{1}{2} \\ \frac{t}{2} + \frac{1}{4} & \text{for } \frac{1}{2} \le t \le \frac{3}{4} \\ t - \frac{1}{8} & \text{for } \frac{3}{4} \le t \le \frac{7}{8} \\ 2t - 1 & \text{for } \frac{7}{8} \le t \le 1 \end{cases}$$

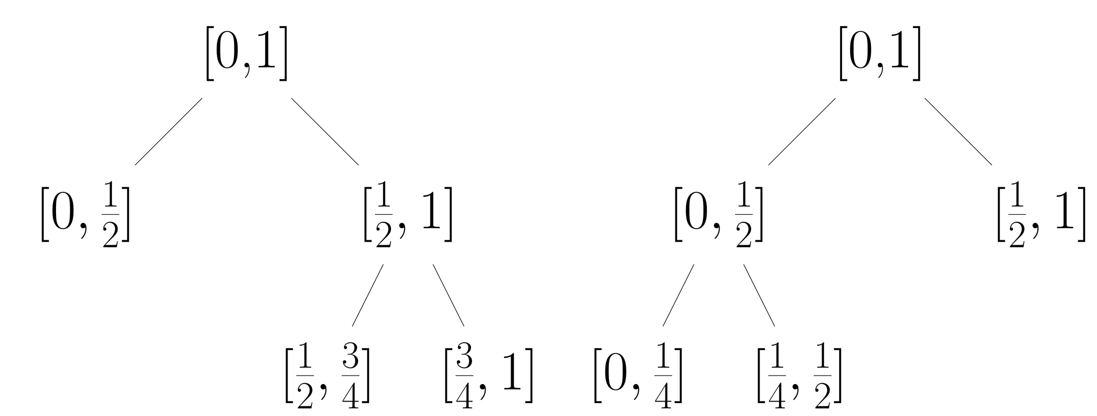
Properties of Thompson's Group

1. **Torsion-free.** For each element x of F, there is no positive integer n for which $x^n = 1$
2. **Subgroups isomorphic to $\mathbb{Z}^k$** F contains a subgroup isomorphic to $\mathbb{Z}^k$ for each positive integer k.
3. **Subgroup isomorphic to $F \times F$** F "contains a copy of $F \times F$.
4. **Finite Presentation** F admits a finite presentation, (which we give in detail later).

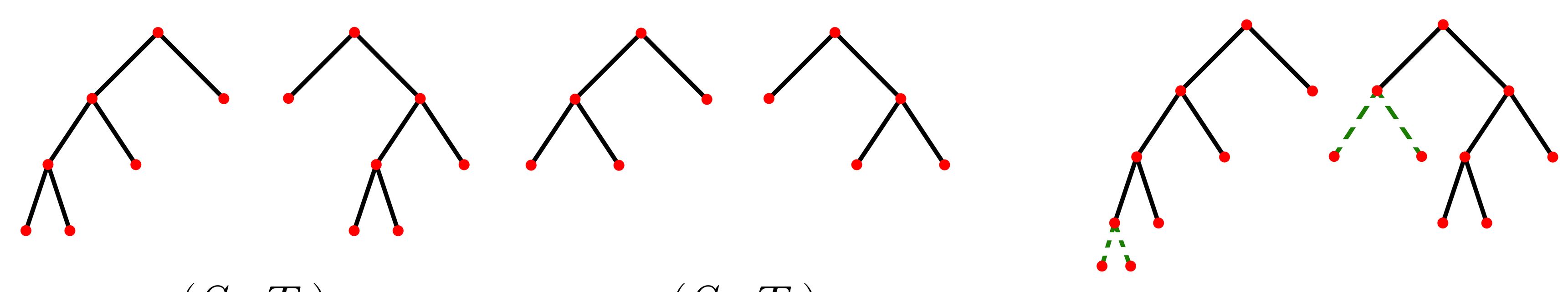
A Combinatorial Definition of Thompson's Group F

The following is an example of a **tree pair diagram** which consists of two trees with the same number of carets. Each tree pair diagram describes the x breakpoints and the y breakpoints of a function in Thompson's group. For example, the function with breakpoints $\{0, \frac{1}{2}, \frac{3}{4}, 1\}$ and $\{0, \frac{1}{4}, \frac{1}{2}, 1\}$ can be seen with the following tree pair diagram where the first tree in the pair is our *source tree* S and the second is our *target tree* T:

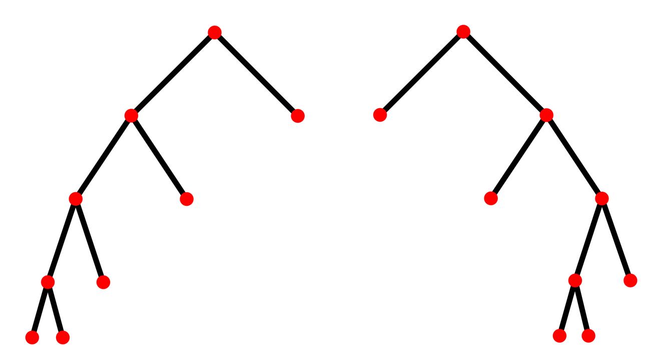
$$x_0(t) = \begin{cases} \frac{t}{2} & \text{for } 0 \le t \le \frac{1}{2} \\ t - \frac{1}{4} & \text{for } \frac{1}{2} \le t \le \frac{3}{4} \\ 2t - 1 & \text{for } \frac{3}{4} \le t \le 1 \end{cases}$$



Analytically, we can combine two elements of Thompsons group via function composition. Combinatorially, we can take two tree pair diagrams that represent elements of Thompson's group and combine them to produce a new tree pair diagram as follows:



(S_1, T_1) (S_2, T_2) (S_1', T_1') (S_2', T_2')



$(S_1, T_1) \cdot (S_2, T_2) = (S_1', T_2')$

Presentations for Thompson's Group F

First, we give an infinite presentation for F. $\langle x_0, x_1, \dots \mid x_i x_j x_i^{-1} = x_{j+1}, \quad i < j \rangle$
Surprisingly, there is a finite presentation for F given by $\langle x_0, x_1 \mid x_0 x_1 x_0^{-1} = x_2,\ x_1 x_2 x_1^{-1} = x_3 \rangle$

1. Lower-index generators conjugate higher-index generators to the next generator.
2. By conjugating x_1 by x_0 we can get to every other generator in the infinite generating set.
3. Given $r_i, s_i > 0$, every element of F can be put in normal form as follows:
$$x_{i_1}^{r_1} x_{i_2}^{r^2} \cdots x_{i_k}^{r_k} x_{j_l}^{-s_l} \cdots x_{j_2}^{-s_2} x_{j_1}^{-s_1}$$
where $i_1 < i_2 < \cdots i_k$ and $j_1 < j_2 < \cdots j_k$.
4. Using *leaf node exponent* of a tree, one can associate to any tree pair diagram, a word in normal form with respect to the infinite presentation.
5. ④ allows us to go from group elements with respect to our combinatorial definition of F to elements of F with repsect to the infinite presentation. ① and ② allows us to go from the infinite presentation to the finite one.

Further Consideration

Thompson's group is isomorphic to the group of piecewise-integral homeomorphisms of the unit interval. In this realization, the breakpoints are rational numbers on the projective line, and the subdivisions come from the Farey sequence. We demonstrate this isometry with the following bijection from rational numbers to dyadic rationals.

Continued Fractions and Trees

Suppose we have an interval $(\frac{a}{b}, \frac{c}{d})$. The adjoint of this interval is $\frac{a+c}{b+d}$. To express a rational number $\frac{p}{q} \in (0,1)$ as a binary tree, realize intervals as nodes and subdivisions as children of nodes connected by a single caret. Start with a node representing the projective interval $(\frac{0}{1}, \frac{1}{0})$. For each interval, divide at the adjoint and draw a caret. If $\frac{p}{q}$ is less than the adjoint, move to the left node. If $\frac{p}{q}$ is greater than the adjoint, move to the right node. Terminate this process when $\frac{p}{q}$ is equal to the adjoint. This process creates a series of leaf nodes, which are on either the left or the right side of a tree. Count the lengths of each sequence of consecutive left nodes, and count the lengths of each sequence of consecutive right nodes. Count the bottom caret as a left node if the previous node is left, and as a right node if the previous node is right. Starting from the top of the tree, the length of the n^{th} sequence is the n^{th} digit of the continued fraction representation of $\frac{p}{q}$.

Minkowski's ?(x) function

Minkowski's ?(x) function is a bijective function from the closed unit interval to itself. This function maps quadratic irrational numbers to rational numbers, and rational numbers to dyadic rational numbers. If we have the continued fraction representation of a rational number, we can compute:

$$?([a_0, a_1, \dots, a_n]) = \frac{(-1)^1}{2^{a_1}} + \frac{(-1)^2}{2^{a_2}} + \cdots + \frac{(-1)^n}{2^{a_n}}.$$

Apply the continued fraction algorithm to $?\left(\frac{p}{q}\right)$, but divide at the average instead of the adjoint. This tree is identical to the tree of the pre-image.