Surfaces: Construction, Metrics and Games

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University of California, Santa Barbara | 2020 Mathematics Directed Reading Program

Surfaces

Definition: A *surface* is a metric space X such that every point in X has a neighborhood which is homeomorphic to the plane.

Supplementary Definitions

- A **metric space** is a set X together with a map $d : X \times X \to \mathbb{R}$ satisfying nondegeneracy, symmetry and the triangle property.
  1. *Nondegeneracy.* $d(x, y) \geq 0$ for all x, y, with equality if and only if $x = y$.
  2. *Symmetry.* $d(x, y) = d(y, x)$ for all x, y.
  3. *Triangle Inequality.* $d(x, z) \leq d(x, y) + d(z, y)$ for all x, y, z.
- Let X be a metric space with metric d. An **open ball** in X with center of the ball as c and the radius as r is a subset of the form $\{x \mid d(x, c) < r\}$.
- A subset $U \subset X$ is **open** if for every point $x \in U$ there is some open ball B_x such that $x \in B_x$ and $B_x \subset U$. A **neighborhood** of a point $x \in X$ is any open subset $U \subset X$ such that $x \in U$.
- A map $h : X \to Y$ is a **homeomorphism** if h is a bijection and both h and h^{-1} are continuous. The spaces X and Y are said to be **homeomorphic** if there is some homeomorphism from X to Y.

One popular surface is the **torus**.

Torus

An example of a torus is the set

$$T^2 = \{(x, y, z, w) \mid x^2 + y^2 = 1; z^2 + w^2 = 1\}.$$

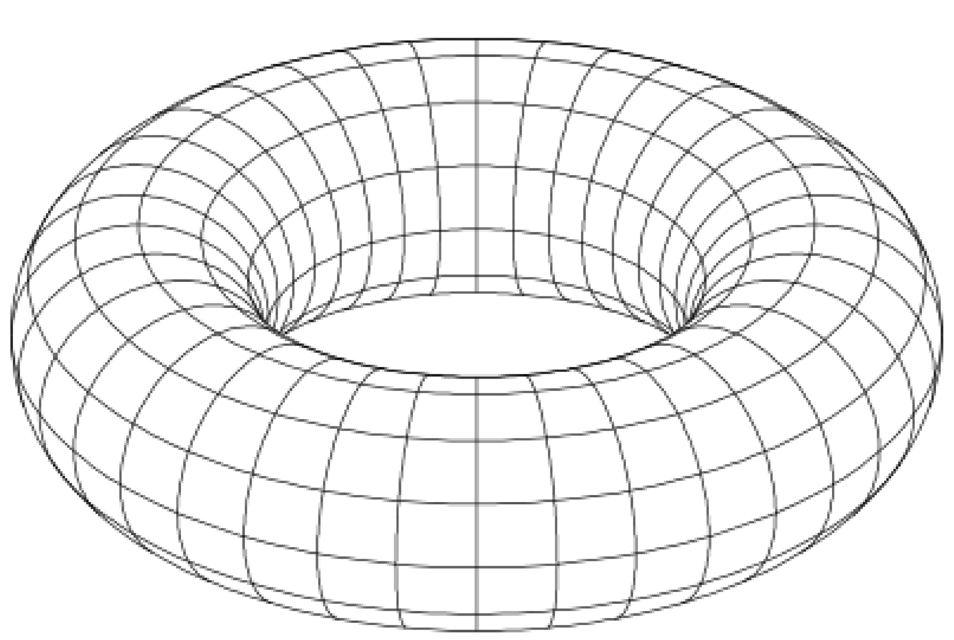

Figure 1: A visualization of a torus.

Construction of the Torus

One way in which we can construct the torus (and other surfaces) is by *gluing* together the edges of polygons. We "glue" together edges by identifying points along the edges with each other.

To create the torus, we begin with a rectangle. Then, we pull the long edges to be adjacent so that we have a cylinder. We finish creating our torus by connecting the ends of the cylinder. In other words, we identify points along the top and bottom edges as equivalent in order to create the cylinder and we mark points along the right and left edges as equivalent to fold the cylinder into the torus.

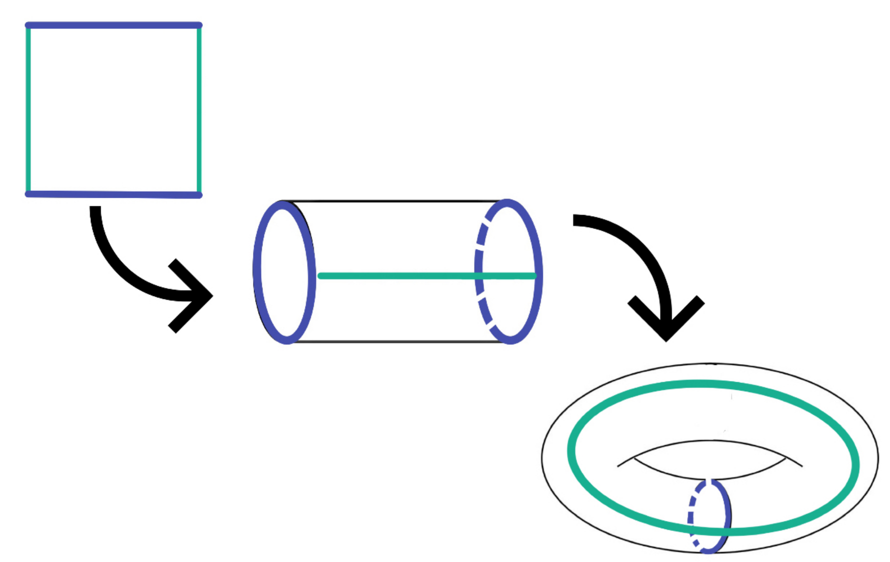

Figure 2: A visualization of the gluing of the torus.

Definitions for Gluing

- An **equivalence relation** on a set X is a relation $\sim$ that satisfies the following properties:
  1. $x \sim x$ for all x
  2. $x \sim y$ if and only if $y \sim x$
  3. $x \sim y$ and $y \sim z$ imply $x \sim z$
- An **equivalence class** of x is the set of all elements which are equivalent to X i.e. the subset $S = \{y \in X \mid y \sim x\}$.
- The **quotient** of X by $\sim$, or the **set of equivalence classes**, is denoted as $X/\sim$.

A New Metric For Glued Surfaces

The **infimum** of a subset A of real numbers is the smallest member of the closure of A, whose existence is guaranteed by the Axiom of Completeness. Now, let X be a set and let $\delta : X \times X \to \mathbb{R}$ be a map that satisfies symmetry and nondegeneracy.

Take $x, y \in X$ and say that a **chain** from x to y is a finite sequence of points $x = x_0, x_1, ..., x_n = y$. We will call this chain C and define

$$\delta(C) = \delta(x_0, x_1) + \delta(x_1, x_2) + ... + \delta(x_{n-1}, x_n).$$

Then, we define the metric $d(x, y) = \inf_C \delta(C)$.

Pathification Metric

The function $d(x, y)$ takes the infimum over all possible chains from x to y. This is sometimes referred to as the *pathification* of δ.

The quotient $X/\sim$ is called a **good quotient** if d, the pathification of δ, is a metric on $X/\sim$.

Another Constructed Surface

Projective Plane

Definition: The projective plane is the quotient of the sphere S^2 by the equivalence relation $p \sim -p$.

We can also construct the projective plane by our gluing construction by gluing opposite pairs of edges of a rectangle together with a half twist.

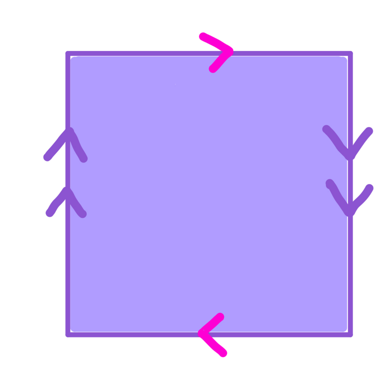

Figure 3: A visualization of the projective plane.

Games on Surfaces

A fun application of this is the exploration of how games on a surface may change due to the properties of the surface. By modifying sudoku to fit on surfaces, we glue edges together, which increases the interaction between rows and columns.

Though we will focus on sudoku on the projective plane, interested readers are encouraged to try various games on surfaces for themselves at https://tinyurl.com/PuzzlesOnSurfaces.

Sudoku on the Projective Plane

Our sudoku board on the projective plane is constructed by taking a standard 16×16 grid and gluing the top to the bottom and the right to the left with a half twist.

1. The boxes must alternate between the numbers 1-16 and the letters A-P.
2. Each row and column must have one instance of each of the numbers 1-16 and A-P.

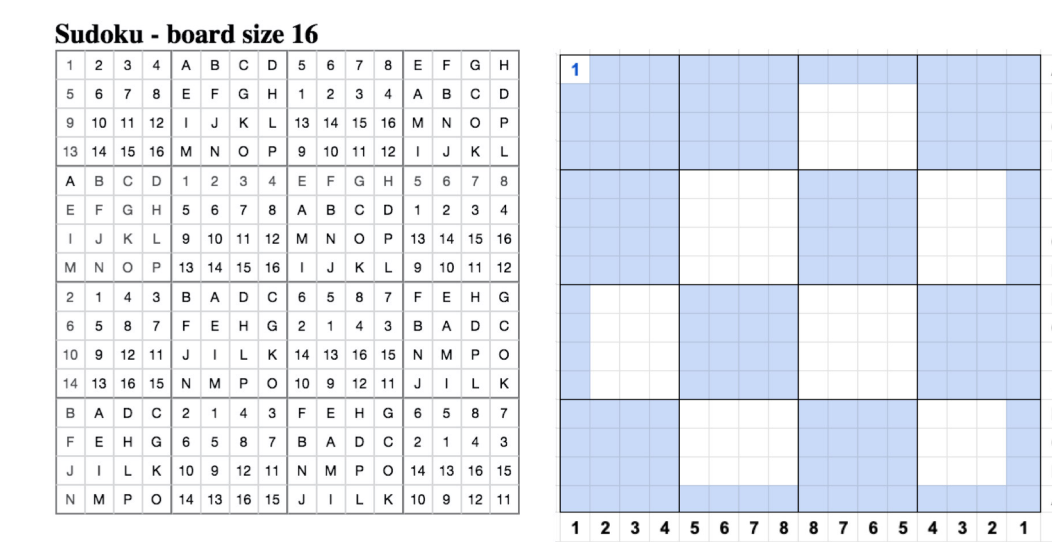

Figure 4: On the left, an example of a valid tiling of sudoku on the projective plane with board size 16. On the right, the blue shading indicates all of the boxes which are now eliminated from containing 1.

A clue on a regular size 16 sudoku board would give you information about the row, column and box of the clue; so, it would give you information about 40 pieces. This is 15.625% informative.

A clue on the projective plane size 16 sudoku board will give you information about longer rows and columns. Additionally, it will indicate which boxes are made up of numbers and letters. A single clue, like the 1 in the image above, gives you information about 25.78% of the remaining board that is to be filled with numbers. A visualization of this idea can be noted in the figure above. So, a clue on projective plane sudoku is more informative than in regular sudoku.

References

- Rosenhouse, Jason and Taalman, Laura. *Taking Sudoku Seriously: The Math Behind The World's Most Popular Pencil Puzzle.* Oxford University Press, New York, New York, 2011.
- Schwartz, Richard E. *Mostly Surfaces.* American Mathematical Society, Providence, Rhode Island, 2011.