Quadratic Forms, The Witt Ring, and Frobenius Reciprocity

Isaiah Martinez and Nathan Schley
University of California, Santa Barbara - Department of Mathematics
2020 Directed Reading Program

Introduction

Through this program I was granted the opportunity to read through some graduate level algebra, some of which shall be described here. It should be noted that any field referenced within this poster should be taken to be a field with characteristic different from 2. We shall begin with some definitions.

Definitions

A *quadratic form* over a field $\mathbb{F}$ is a polynomial f in n variables over $\mathbb{F}$ that is homogeneous of degree 2. We may customarily write f as

$$f(X) = \sum_{i,j} \frac{1}{2}(a_{ij} + aji)X_iX_j = \sum_{i,j} a'_{ij}X_iX_j.$$

Moreover, f determines a unique symmetric matrix (a'_{ij}), denoted by M_f. With regards to matrix notations, we have

$$f(X) = (X_1, \dots, X_n) \cdot M_f \cdot \begin{pmatrix} X_1 \\ \vdots \\ X_n \end{pmatrix} = X^T \cdot M_f \cdot X.$$

Let V be any finite-dimensional $\mathbb{F}$-vector space, and $B : V \times V \to \mathbb{F}$ a symmetric bilinear pairing on V. We shall all the pair (V, B) a *quadratic space*, and associate with it a *quadratic map*, $q = q_B : V \to \mathbb{F}$, given by $q(x) = B(x, x); (x \in V)$.

For any $d \in \mathbb{F}$, we denote the isometry class of the 1-dimensional space corresponding to the quadratic form dX^2 by $\langle d \rangle$. It is worth noting that $\langle d \rangle$ is regular if and only if $d \in \dot{\mathbb{F}}$. With regards to notation, we abbreviate the diagonal form $\langle d_1 \rangle \perp \dots \perp \langle d_n \rangle$ as $\langle d_1, \dots, d_n \rangle$.

Let v be a nonzero vector in a quadratic space (V, B). We call v an isotropic vector if $B(v, v) = 0$ and say that v is *anisotropic* otherwise. The quadratic space (V, B) is said to be *isotropic* if it contains a nonzero isotropic vector, and is said to be *anisotropic* otherwise.The isometry class of a 2-dimensional quadratic space is called the *hyperbolic plane* if it satisfies the following conditions:

1. V is regular and isotropic, with $d(V) = -1 \cdot \dot{\mathbb{F}}^2$.
2. V is isometric to $\langle 1, -1 \rangle$.
3. V corresponds to the equivalence class of the binary quadratic form X_1X_2.

The hyperbolic plane will be denoted by $\mathbb{H}$.

A quadratic form or quadratic space is called *universal* if it represents all the nonzero elements of $\mathbb{F}$.

It is worth noting as well that any quadratic space (V, q) splits into an orthogonal sum

$$(V_t, q_t) \perp (V_h, q_h) \perp (V_a, q_a),$$

where V_t is totally isotropic, V_h is hyperbolic, and V_a is anisotropic. This splitting is unique up to isometry.

The *Witt-Grothendieck ring* $\widehat{W}(\mathbb{F})$ is the ring generated by the monoid of isometry classes with $V_t = \{\vec{0}\}$ under $\perp$ with formal inverses, and multiplication given by $\underset{\otimes}{\mathbb{F}}$. The factor ring $W(\mathbb{F}) = \widehat{W}(\mathbb{F})/\mathbb{Z} \cdot \mathbb{H}$ is called the *Witt Ring* of $\mathbb{F}$. Using the usual procedure of extension of scalars, it is easy to verify that $\widehat{W}$ and W are both "functors" from fields to commutative rings.

The Frobenius Reciprocity Theorem

For every field extension of $\mathbb{F}$ there is a corresponding extension of scalars for $\widehat{W}(\mathbb{F}) \to \widehat{W}(K)$. There is a commonly studied transfer map $s_* : \widehat{W}(K) \to \widehat{W}(\mathbb{F})$ when $[K : \mathbb{F}] < \infty$. This theorem relates the two.

Let K be a finite extension of the field $\mathbb{F}$. Given an $\mathbb{F}$-quadratic space (V, B, q), we can construct a K-quadratic space (V_K, B_K, q_K) by the extension of scalars. The underlying space V_K is taken to be $K \otimes_{\mathbb{F}} V$, and B_K is taken to be the unique symmetric bilinear form on V_K such that

$$B_K(k \otimes v, k' \otimes v') = kk'B(v, v') \qquad (k, k' \in K; v, v' \in V).$$

We denote the monoid of all isometry classes of quadratic forms over $\mathbb{F}$ respectively over K under the operation $\perp$ as $M(F)$ (resp. $M(K)$). The construction $q \mapsto q_K$ above induceces a monoid homomorphism $M(\mathbb{F}) \to M(K) \subseteq \widehat{W}(K)$ which then induces a well defined, ring homomorphism $W(\mathbb{F}) \to W(K)$. We name the inclusion map $\mathbb{F} \subseteq K, r$, and we can write $\hat{r}^*$ and r^* to denote the maps $\widehat{W}(\mathbb{F}) \to \widehat{W}(K)$ and $W(\mathbb{F}) \to W(K)$ (resp.) given by $\hat{r}^*(V) = V_K$.

Now let $s : K \to \mathbb{F}$ be a nonzero $\mathbb{F}$-linear functional on the $\mathbb{F}$-vector space K. Then for any K quadratic space (U, B), we may compose the pairing $B : U \times U \to K$ with the functional s to get an $\mathbb{F}$-bilinear pairing

$$sB : U \times U \to \mathbb{F}.$$

Thus, the K-quadratic space (U, B) gives rise to an $\mathbb{F}$-quadratic space (U, sB). We can write $s_*(U)$ to denote the quadratic space U over $\mathbb{F}$ with the bilinear form sB. We call $s_*(U)$ the "transfer" of U, noting that

$$\dim_{\mathbb{F}} s_*(U) = [K : \mathbb{F}] \cdot \dim_K U.$$

This construction can be applied to the 1-dimensional K-space K that carries the natural bilinear form $(x, y) \mapsto xy \quad (x, y \in K)$. As a K-quadratic space, this is simply $\langle 1 \rangle_K$.

Finally, if V is a quadratic space over $\mathbb{F}$ and U is a quadratic spcae over K, then there exists an $\mathbb{F}$-isometry

$$s_*((r^*V) \otimes_K U) \cong V \otimes_{\mathbb{F}} s_*(U).$$

In particular, if $U = \langle 1 \rangle_K$, we have

$$s_*(r^*V) \cong V \otimes_F s_*(\langle 1 \rangle_K).$$

Proof

We begin by defining a map

$$f : s_*((K \otimes_{\mathbb{F}} V) \otimes_K U) \to V \otimes_{\mathbb{F}} s_*(U)$$

by $(k \otimes_{\mathbb{F}} v) \otimes_K u \mapsto v \otimes_{\mathbb{F}} (ku)$. This is an $\mathbb{F}$-isomorphism with an inverse given by $v \otimes u \mapsto (1) \otimes u$. We may claim that f is the required isometry, and it suffices to show that it is.

Take $k, k' \in K, v, v' \in V$, and $u, u' \in U$. With respect to $V \otimes_F s_*(U)$ we have,

$$\langle f((k) \otimes u), f((k' \otimes v') \otimes u' \rangle = \langle v \otimes (ku), v' \otimes (k'u') \rangle$$

$$= \langle v, v' \rangle_V \cdot \langle ku, k'u' \rangle_{s_*U} = \langle v, v' \rangle_V \cdot s(kk'\langle u, u' \rangle_U).$$

Conversely, calculating in $s_*((K \otimes_{\mathbb{F}} V) \otimes_K U)$, we get

$$\langle (k) \otimes u, (k' \otimes v') \otimes u' \rangle_{s_*(\dots)} = s(\langle k \otimes v, k' \otimes v' \langle_{K \otimes V} \langle u, u' \rangle_U)$$

$$= s(kk', v' \rangle_V \cdot \langle u, u' \rangle_U) = \langle v, v' \rangle_V \cdots (kk' \langle u, u' \rangle_U),$$

since $\langle v, v' \rangle_V \in \mathbb{F}$, and s is $\mathbb{F}$-linear. This proves that f is an $\mathbb{F}$-isometry.

■

Corollaries

1. If U is a hyperbolic space over K, then $s_*(U)$ is a hyperbolic space over $\mathbb{F}$.

2. $U \mapsto s_*(U)$ defines group homomorphisms

$$s_* : \widehat{W}(K) \to \widehat{W}(\mathbb{F}) \quad \text{and} \quad s_* : W(K) \to W(\mathbb{F}).$$

3. The composition

$$\widehat{W}(\mathbb{F}) \xrightarrow{\hat{r}^*} \widehat{W}(K) \xrightarrow{s_*} \widehat{W}(\mathbb{F})$$

coincides with the multiplication by $s_*(\langle 1 \rangle_K)$. And the same follows for W.

4. $\operatorname{im}(s_*)$ is an ideal in $\widehat{W}(\mathbb{F})$, and the same follows for W.

5. Let $T \subseteq W(\mathbb{F})$ be the transfer ideal for a finite field extension $K/\mathbb{F}$, and let $W(K/\mathbb{F})$ denote the kernel of $r^* : W(\mathbb{F}) \to W(K)$. Then $T \cdot W(K/\mathbb{F}) = 0$.

Proof

1. Since $s_*(U_1 \perp U_2) \cong s_*(U_1) \perp (U_2)$, it suffices to show the case $U = \mathbb{H}_K$. Using the last statment in the Frobenius Reciprocity theorem, we get

$$s_*(U) = s_*(\mathbb{H}_K) = s_*(\hat{r}^*(\mathbb{H}_F)) \cong \mathbb{H}_{\mathbb{F}} \otimes_{\mathbb{F}} s_*(\langle 1 \rangle_K).$$

The last quadratic space is isomorphic to $[K : \mathbb{F}] \cdot \mathbb{H}_{\mathbb{F}}$, as desired.

The proofs of 2, 3, and 4 follow from the Frobenius Reciprocity Theorem and the first Corollary.

5. We can fix any nonzero $\mathbb{F}$-linear functional $s : K \to \mathbb{F}$, we can take T to be $\operatorname{im}(s_* : W(K) \to W(\mathbb{F}))$. For wny K-quadratic space U and $\mathbb{F}$-quadratic space V, Frobenius Reciprocity givves us

$$s_*(V_K \otimes_K U) \cong V \otimes_{\mathbb{F}} s_*(U).$$

If V_K is hyperbolic, then the left hand side is hyperbolic by 1.4. In this case, the above isomorphics shows that V is annihilated by T in $W(\mathbb{F})$.

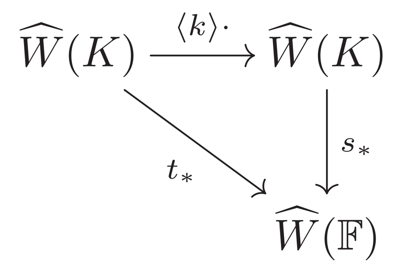
■

Concluding Remarks

We may note that $s_* : \widehat{W}(K) \to \widehat{W}(\mathbb{F})$ is a morphism of $\widehat{W}(\mathbb{F})$-modules, if we view $\widehat{W}(K)$ as a $\widehat{W}(\mathbb{F})$-module via the ring homomorphism $\hat{r}^* : \widehat{W}(\mathbb{F}) \to \widehat{W}(K)$. We also get the same for W. This reveals a nice way of interpreting the Frobenius Reciprocity Law.

Although s_* is not a ring homomorphism, it is nevertheless "functorial." Moreover, if $\mathbb{F} \subseteq K \subseteq L$ is a tower of finite extensions, and if $s : K \to \mathbb{F}$ is $\mathbb{F}$-linear and $t : L \to K$ is K-linear (both of which being nonzero), then it follows that $(s \circ t)_* = s_* \circ t_*$.

It is natural to ask to what extent $s_* : \widehat{W}(K) \to \widehat{W}(\mathbb{F})$ depend on the choice of the nonzero $\mathbb{F}$-linear functional s. Since $s_*(\langle 1 \rangle_K)$ is a regular $\mathbb{F}$-quadratic space by 1.1, every $\mathbb{F}$-linear functional on K is of the form $z \mapsto s(kz)$ for some $k \in K$. Thus, for any nonzero $t : K \to \mathbb{F}$, there exists a commutative diagram



where $k \in \dot{K}$ is such that $t(z) = s(kz)$ for all $z \in K$. This says that s_* and t_* are the same up to a group automorphism of $\widehat{W}(K)$ given by multiplication by the unary form $\langle k \rangle_K$. In particular, the ideal $s_*(\widehat{W}(K))$ is independent for the choice of s; it may be called the transfer ideal with respect to $K/\mathbb{F}$, and we see the same again for W.

Acknowledgments

I would like to thank Nathan Schley, for his tremendous amount of help with this project, and throughout the DRP program altogether. I have really enjoyed this experience, and Nathan's mentorship and guidance means the world to me. I would also like to thank the UCSB Math Department for this very rewarding opportunity. Finally, I would like to thank my family, who give me the love, support, and strength to pursue mathematics.

References

Lam, T. Y. Introduction to Quadratic Forms over Fields. American Mathematical Society, 2005.