Understanding the McKay Correspondence

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Representation Theory

When we have a problem in algebra, we would like to be able to reduce the problem to linear algebra. Representation theory provides the necessary tools for such simplification.

Let G be a finite group and V a vector space over F that G acts on. We can define a homomorphism

$$\phi : G \to \mathrm{GL}(V),$$

which is called a *linear representation* of G. Furthermore, if $n = \dim(V)$, fixing a basis $\mathrm{GL}(V) \cong \mathrm{GL}_n(F)$, giving a matrix representation of G.

For a subspace W of V, (ϕ_W, W) is a subrepresentation of (ϕ, V) if the action of G stabilizes W (i.e. if $g \in G$ and $x \in W$, $g \cdot x \in W$.) We define irreducible representations (irreps) to have no subrepresentations other than itself and $\{0\}$.

Lemma: (Schur's lemma) An equivariant map (i.e. a map that commutes with action of G onto the representation) between two irreps is either an isomorphism or the zero map.

A representation can be determined by its character, the trace of its matrix representation. Given two representations V, W and the corresponding characters χ_V, χ_W, we use the inner product

$$\langle \chi_V, \chi_W \rangle = \frac{1}{|G|} \sum_{g \in G} \chi_V(g) \overline{\chi_W(g)}$$

to get the following relation

$$\langle \chi_V, \chi_W \rangle = \begin{cases} 1 \text{ if } V \cong W \\ 0 \text{ if } V \ncong W \end{cases}$$

and for $V = W^a \oplus W'$,

$$\langle \chi_V, \chi_W \rangle = a \qquad (1)$$

References

[1] David A. Cox. *Lectures on Toric Varieties*, 2005.

[2] Joe Harris and William Fulton. *Representation Theory: A First Course* Graduate Texts in Mathematics, Volume 129, Springer-Verlag, 1991.

[3] Ravi Vakil. *The Rising Sea* http://math.stanford.edu/ vakil/216blog/FOAGnov1817public.pdf

McKay Correspondence

The McKay graph of a finite subgroup G of SU_2 constructed from the representations of G is the same as the graph constructed from the minimal resolution of singularities of $\mathbb{C}^2/G$.

Representation Theory for McKay

From the representation theory perspective, the McKay graph is a directed graph constructed by looking at tensors of irreducible representations with the inclusion map $i : G \to GL(\mathbb{C}^2)$. Let $\rho_0, \ldots, \rho_r$ be the irreducible representations of G. Then the vertices of G be $\{\rho_0, \ldots, \rho_r\}$. We draw an edge $\rho_j \to \rho_k$ if ρ_k is a summand of $\rho_j \otimes i$.

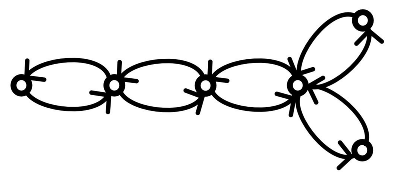


Figure 1:Graph for BD_n

The character of $\rho \otimes \mu$ is $\chi_\rho \chi_\mu$ for representations ρ, μ. Using equation 1 allows us to easily find the McKay graph since the character tables of finite groups are completely found. As in the picture, all of the graphs will be dynkin diagrams with $2-$cycles for edges.

Algebraic Geometry for McKay

Take the variety $\mathbb{C}^n/G = X$. There is a polynomial $p(x)$ such that the variety $p(x) = 0$ is isomorphic to X. We resolve the singularities, points where all the partials. This amounts to passing to projective space where we can remove a singularity. This process called a blow up gives a copy of $\mathbb{P}^1$.

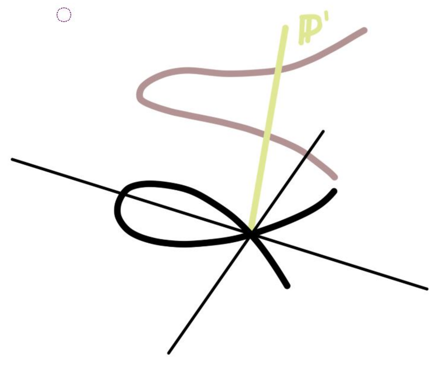
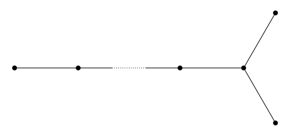
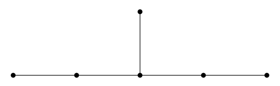
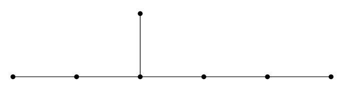
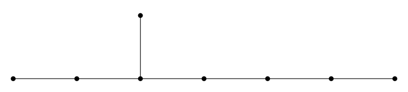


Figure 2:Resolving singularity for A_2

We draw a vertex for each copy of $\mathbb{P}^1$ and draw an edge between them if they intersect. When $G = A_n$, the variety is toric. *Using the theory of Neron Models of elliptic surfaces and the semi stable reduction theorem, one can reduce the study of D_n and E_6, E_7, E_8 singularities to An singularities...*

McKay Correspondence Chart

Subgroup of SU_2	Polynomial	Graph	Name of Graph
$\mathbb{Z}/(n+1)$ cyclic	$x^2 + y^2 + z^{n+1}$		A_n
$BD_{4(n-2)}$ binary dihedral	$x^2 + y^2 z + z^{n-1}$		D_n
binary tetrahedral	$x^2 + y^3 + z^4$		E_6
binary octahedral	$x^2 + y^3 + yz^3$		E_7
binary iscosahedral	$x^2 + y^3 + z^5$		E_8

Toric Varieties

Definition: An irreducible variety is toric if

- $(\mathbb{C}^\star)^n$ is a Zariski open subset of X, which we call the torus
- the action of $(\mathbb{C}^\star)^n$ on itself extends to an action of $(\mathbb{C}^\star)^n$ on X

Examples: $\mathbb{C}^n$ with torus $(\mathbb{C}^\star)^n$, $\mathbb{P}^n$ using the image under $(t_1, \ldots, t_n) \to (1, t_1, \ldots, t_n)$.

Let $N \cong \mathbb{Z}$, $M = \mathrm{Hom}_{\mathbb{Z}}(N, \mathbb{Z})$. The toric variety from a strongly convex rational polyhedral cone $\sigma = Cone(S)$ for $S \subset N$ finite uses the dual cone $\sigma^\vee = \{u \in \mathbb{R}^{n\star} | \langle u, v \rangle \geq 0 \text{ for all } v \in \sigma\}$.

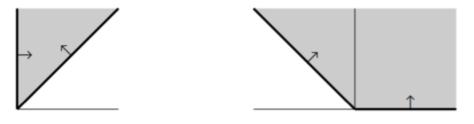


Figure 3:$\sigma = Cone(e_1 + e_2, e_2)$ and its dual from [1]

Definition: Let $S_\sigma = \sigma^\vee \cap M$. Then the affine toric variety associated to σ is

$$V_\sigma = Spec(\mathbb{C}[S_\sigma])$$

Fact: An affine toric variety is isomorphic to V_σ for some σ if and only if V is normal.

From a fan Σ, a finite collection of strongly convex rational polyhedral cones that agree on intersection, construct another toric variety by gluing.

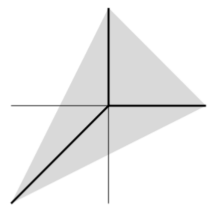


Figure 4:Fan using $e_1, e_2, -e_1 - e_2$ for $\mathbb{P}^2$ from [1]

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