# Topological Graph Theory

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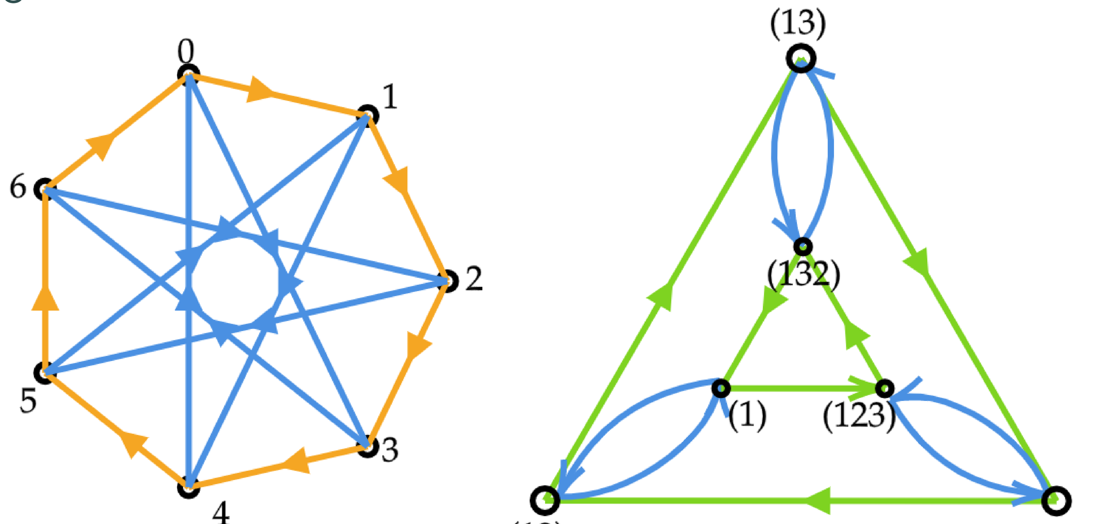
## Overview

Topological graph theory is a combination of topology, group theory, and graph theory. By looking at the overlaps of these areas, mathematicians have gotten new insights into each individual area.

## Graphs

A **Cayley graph** represents a group. It has the group's elements as its vertices, and its edges connect vertices related by group multiplication with one of the elements of the chosen generator.




The cyclic group $\mathcal{Z}_7$ with a generating set {1, 3}

The symmetric group $S_3$ with a generating set {(1 2 3), (1 2)}

**Scheier Coset Graphs** are generalizations of Cayley graphs. Their vertices represent cosets of groups

Let $G$ be a group, with a subgroup $H$, and a generating set $A = \{A_1, ..., A_n\}$. Then a **left coset** of $H$ in $G$ is $dH = \{dh | h \in H\}$ where $d$ is any element in $G$.

The vertices of this graph are the left cosets of $H$ in $G$. There are edges between a coset and the cosets generated by multiplying it with a generator.

$G = S_3 = \{(), (1\,2), (2\,3), (1\,3), (1\,2\,3), (1\,3\,2)\}$

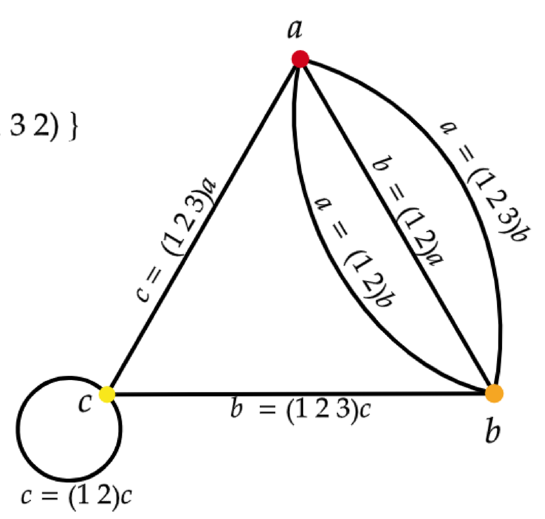
$H = \{(), (1\,3)\}$

$A = \{(1\,2), (1\,2\,3)\}$

• $a = ()H = (1\,3)H = \{(), (1\,3)\}$

• $b = (1\,2)H = (1\,3\,2)H = \{(1\,2), (1\,3\,2)\}$

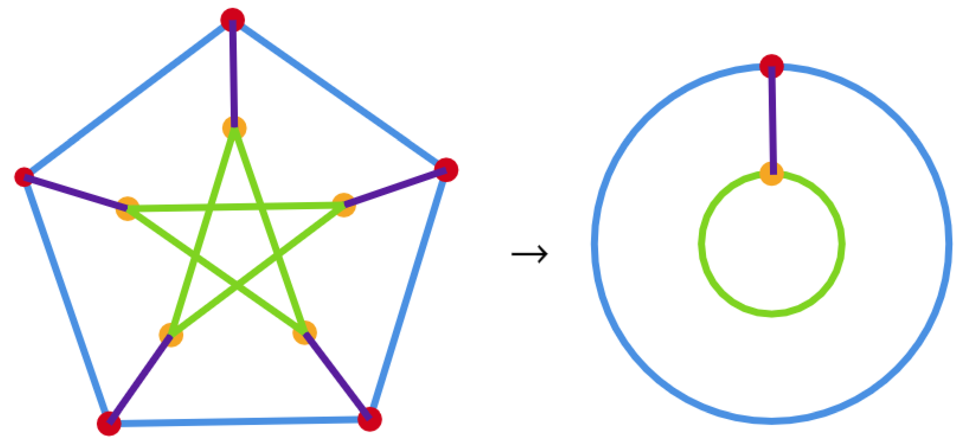
• $c = (2\,3)H = (1\,2\,3)H = \{(2\,3), (1\,2\,3)\}$




A **Quotient Graph** can be made from highly symmetrical graphs.

If we have a graph $G$ and a group $\beta$ whose elements $b \in \beta$ represent a graph automorphism of $G$, $\phi_b : G \to G$, then we can construct a quotient graph.

The set of vertices that can be mapped to each other in a graph automorphism are called "vertex orbits". Edges are denoted similarly. Quotient graphs have as vertices the set of vertex orbits, and as edges the set of edge orbits.

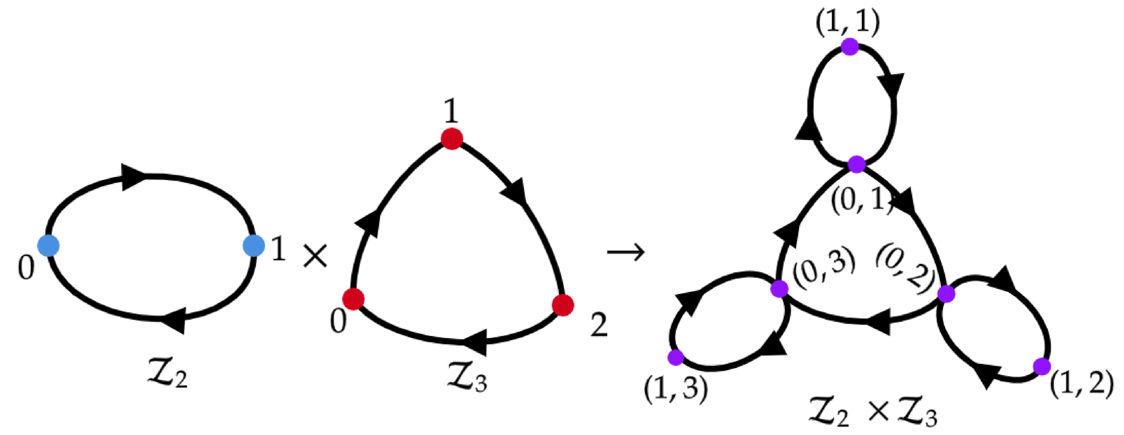


The "Petersen graph"

The same graph after quotienting out $\mathcal{Z}_5$

### Cartesian Product

Some more complex graphs can be represented by the cartesian product of simpler graphs. For example, every Cayley graph representing an abelian group can be written as the cartesian product of cyclic groups.

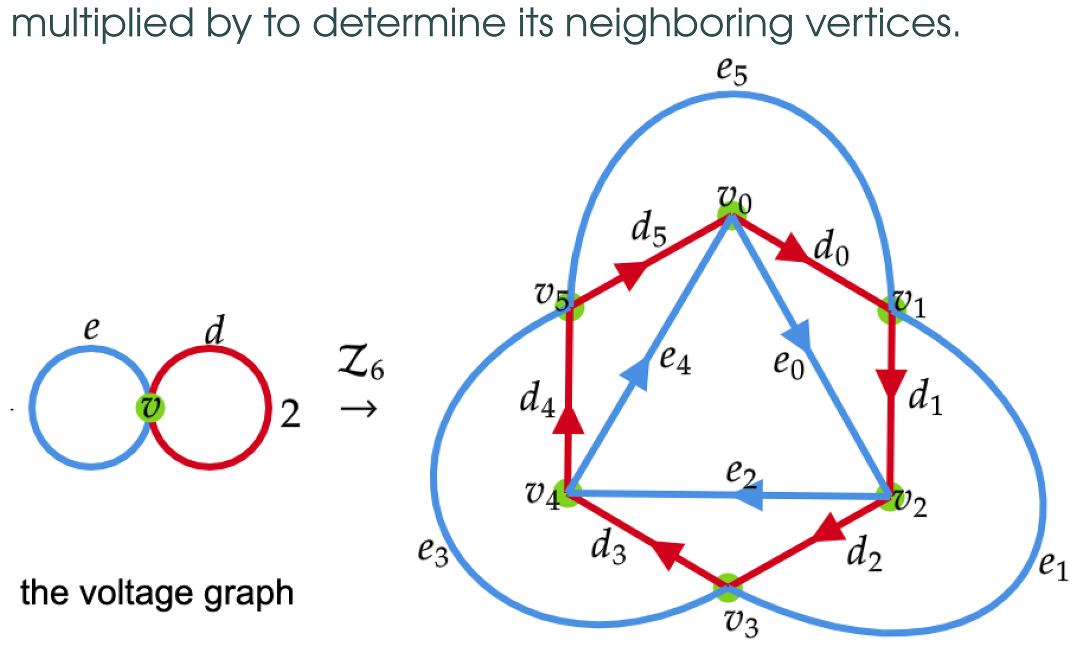



The cartesian product of two graphs has vertex set $V_1 \times V_2$ and edge set $E_1 \times E_2$

A **Voltage Graph** allows you to obtain a complicated, symmetrical graph from a relatively simple one.

Say the initial graph has an edge and vertex set $E$ and $V$. Let $A$ be a group.

The edge set of the derived graph is then $E \times A$ and the vertex set is $V \times A$. A voltage is assigned to each edge of the voltage graph, which dictates the element of $A$ the original vertex is multiplied by to determine its neighboring vertices.

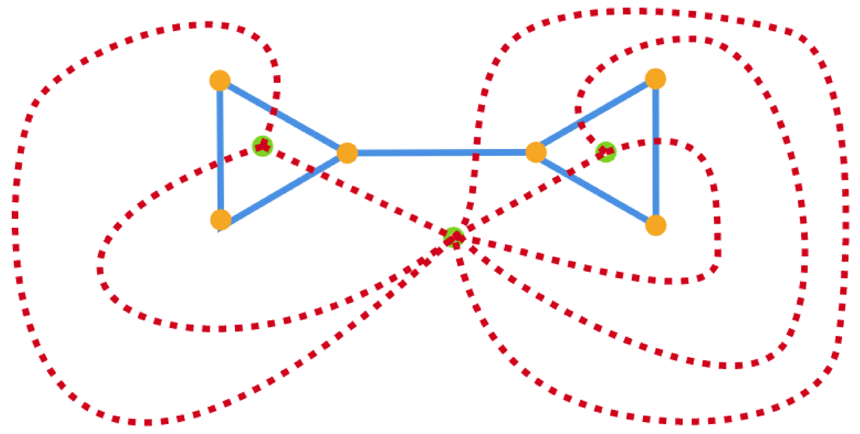



the voltage graph

the derived graph, the 3-octahedron graph

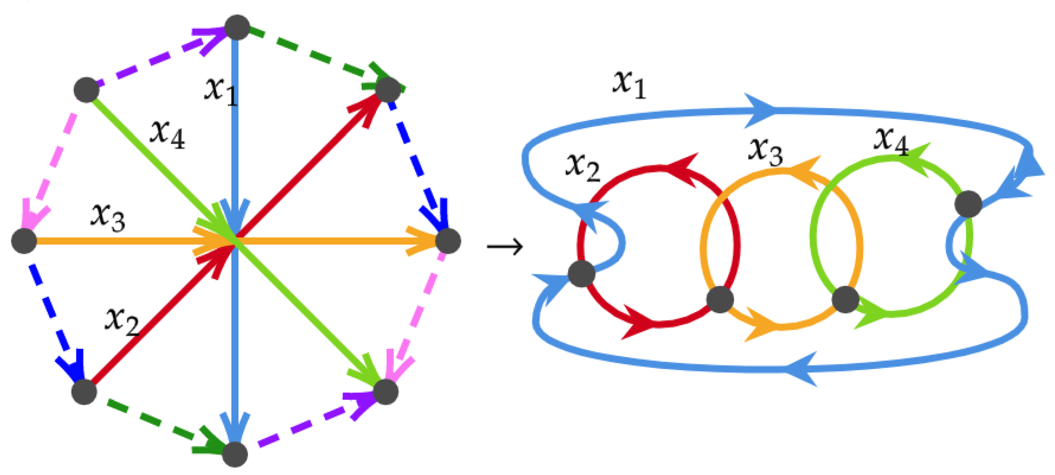
The voltage graph is a **covering** of the derived graph, meaning it is locally isomorphic to the derived graph, i.e. the neighborhoods of the equivalent vertices in the voltage and derived graphs are the same

To construct a **dual graph**, place a vertex inside each face of the original graph and connect the new vertices in adjacent faces with edges.



A **Current graph** is the dual of a voltage graph. They are useful when embedding a graph into a surface

From current graphs, can get the derived embedding, $G_\beta \to S_\beta$, helps with finding the minimum genus of a surface




A voltage graph

Its dual current graph

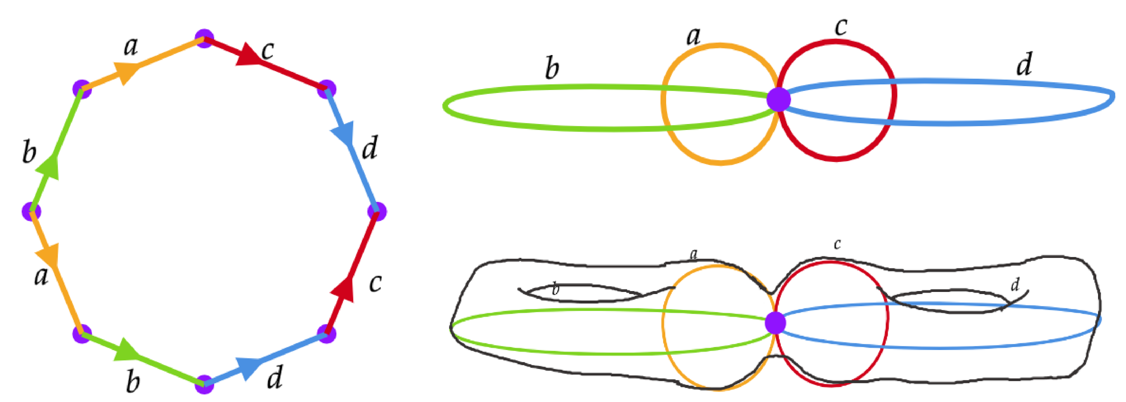
## Representing Surfaces

A surface is **orientable** if a direction can be defined on the surface that doesn't change while traversing the surface.

Cellular embeddings of these surfaces have an **Euler characteristic** $\chi = \#V - \#E + \#F$

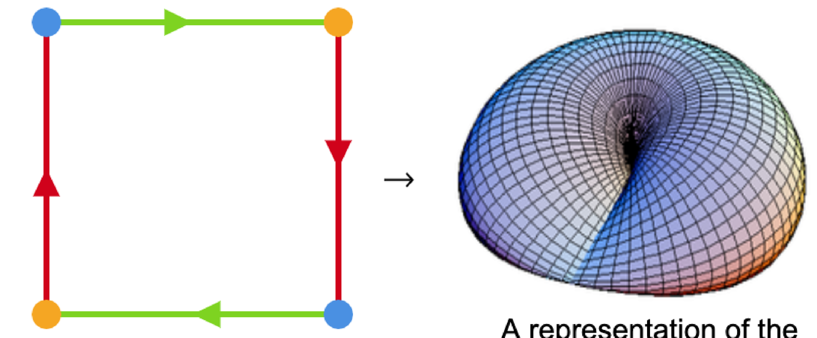
If the surfafce is orientable, its **genus** is $g = 1 - \frac{\chi}{2}$.

If the surface isn't orientable its **crosscap number** is $k = 2 - \chi$




This octogon represents a genus 2 orientable surface.

For the above surface, $\chi = 1 - 4 + 1 = -2$. This confirms that the genus is $g = 1 - \frac{-2}{2} = 2$




A polygonal representation of the projective plane.

A representation of the projective plane

For the above surface, $\chi = 2 - 2 + 1 = 1$. It is non-orientable, so its crosscap number is $k = 2 - 1 = 1$.

Generally, surfaces can be constructed by combining tori, projective planes, and spheres.

## Embedding Graphs into Surfaces

Knowing which graphs are embeddable in which surfaces is very important in topological graph theory.

A graph can be embedded into a surface if it doesn't cross itself on the surface

People are mostly concerned with the **minimum genus**, $\gamma(G)$, a surface must be for a graph to be embedded in it.

This is similar for non-orientable surfaces and crosscap numbers.

the **interpolation theorem** says that a graph can be imbedded into every surface with a genus in between its maxiumum and minimum genus, so if we know the maximum and minimum genus of a graph we know all of its possible imbeddings.

The **Betti number** of a graph, $\beta(G) = 1 - \#V + \#E$ relates to the upper and lower bounds of the genus of a surface it can be imbedded in.

## Applications

Representing groups graphically has led to some major insights about the groups- for example, it allowed distance to be defined on a group. This led to the discovery that the relationship between surface area and boundary length is linear in hyperbolic geometry

topological graph theorists are interested in finding the minimum genus a group can be embedded into. For example, the **Jungerman and White Theorem** says that

A lower bound $\gamma(\mathcal{A}) \geq 1 + \#\mathcal{A} \cdot (\frac{r-2}{4})$ is proved for abelian groups certain abelian groups.

## References

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Peifer, David. "Max Dehn and the Origins of Topology and Infinite Group Theory." The American Mathematical Monthly, vol. 122, no. 3, 2015, p. 217.,