Algebras and Modules as Diagrams

Timothy Xu   Mentor: Andres Barei

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University of California - Santa Barbara

Algebras from Diagrams

Representation Theory seeks to understand algebraic objects through their modules. To represent all 'nice' modules over 'nice' algebras using diagrams, we first define how to read a diagram as an algebra. We fix an algebraically closed field K with characteristic 0.

An **algebra** A is a vector space (over some field K) with a ring structure (with a multiplicative identity).

A **(right) A-module** M is a vector space (over the same field K) with a compatible right action from the algebra A.

A **quiver** Q is an oriented multigraph allowing loops.

The **path algebra** KQ of a quiver Q is an algebra over some field K, with a K-vector space basis consisting of all paths in Q, including those of length 0 (which are in bijection with the vertices of Q). The ring structure on KQ is extended from the following: if the ending vertex of a path matches the starting vertex of another path, their product is defined to be the joined path. Otherwise, their product is 0.

The path algebra of the following quiver has an infinite K-basis

$$\{\epsilon_1, \epsilon_2, \epsilon_3, \alpha, \beta, \delta, \gamma, \beta\gamma, \beta\delta,$$
$$\alpha\beta, \alpha^2\beta, \dots, \alpha\beta\gamma, \alpha^2\beta\gamma, \dots, \alpha\beta\delta, \alpha^2\beta\delta, \dots\}.$$

$$\begin{tikzcd} 1 \arrow[loop above, "\alpha"] \arrow[r, "\beta"] & 2 \arrow[r, bend left, "\gamma"] \arrow[r, bend right, "\delta"'] & 3 \end{tikzcd}$$

To represent all algebras clearly we need some relations to make a path algebra finite-dimensional.

The **radical** of an algebra is the intersection of all maximal right ideals of that algebra. For the path algebra of a finite connected acyclic quiver Q, the radical of KQ $R_Q := \operatorname{rad} KQ$ is generated (as a module) by all the arrows (i.e. paths of length 1) of Q.

A two-sided ideal I of KQ is said to be **admissible** if $\exists m \geq 2$ with $R_Q^m \subseteq I \subseteq R_Q^2$. Then the quotient KQ/I is called a **bound quiver algebra**.

We usually define admissible ideals in terms of generators. A **relation** in Q is a K-linear combination of paths of length at least two, sharing the same source and target. For example in the quiver above, $\beta\gamma - \beta\delta$ is a relation, and α^2 is a relation. If $I := \langle \beta\gamma - \beta\delta, \alpha^2 \rangle$, then KQ/I has a (now finite) K-basis $\{\epsilon_1, \epsilon_2, \epsilon_3, \alpha, \beta, \gamma, \delta, \beta\gamma, \alpha\beta, \alpha\beta\gamma\}$.

Diagrams from Algebras

Now we show how to construct a diagram corresponding to a 'nice' algebra.

An element e of an algebra is an **idempotent** if it satisfies $e^2 = e$. Idempotents e_1, e_2 are **orthogonal** if $e_1 e_2 = e_2 e_1 = 0$. An idempotent is **primitive** if it is not the sum of two nonzero orthogonal idempotents. A set of primitive, pairwise orthogonal idempotents $\{e_1, e_2, \dots, e_n\}$ is **complete** if its sum is $1 \in A$, in which case $A \simeq e_1 A \oplus e_2 A \oplus \dots \oplus e_n A$. We say A is **basic** if $e_i A \not\simeq e_j A$ when $i \neq j$, and A is **connected** if A is not a direct product of two algebras.

Given any basic connected finite-dimensional K-algebra A, the **quiver of A** Q_A is defined as follows: the vertices of Q_A are in bijection with a complete set of primitive orthogonal idempotents $\{e_1, e_2, \dots, e_n\}$; given a pair of vertices (a, b), the arrows between a and b are in bijection with the elements in a K-basis of the algebra $e_a(\operatorname{rad} A/\operatorname{rad}^2 A)e_b$.

Consider the following algebra A over K:

$$A := \left\{\begin{bmatrix} K & 0 & 0 \\ K & K & 0 \\ K & 0 & K \end{bmatrix}\right\}, \operatorname{rad} A = \begin{bmatrix} 0 & 0 & 0 \\ K & 0 & 0 \\ K & 0 & 0 \end{bmatrix}, \operatorname{rad}^2 A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

It has a complete set of primitive orthogonal idempotents

$$e_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, e_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, e_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

The algebras $e_2(\operatorname{rad} A)e_1$ and $e_3(\operatorname{rad} A)e_1$ have dimensions 1, which implies that all other algebras of the form $e_i(\operatorname{rad} A)e_j$ are 0 as $\operatorname{rad} A$ has dimension 2 over K. So the following diagram depicts the quiver of A.

$$\begin{tikzcd} & 1 & \\ 2 \arrow[ur, "\alpha"] & & 3 \arrow[ul, "\beta"'] \end{tikzcd}$$

Modules from Diagrams

1. If Q is a finite connected quiver, and I is an admissible ideal of KQ, then $Q_{KQ_I} = Q$.
2. If A is a basic connected finite-dimensional K-algebra, there exists an admissible ideal I of KQ_A such that $A \simeq KQ_A/I$.

We claim that all finite-dimensional right modules over a basic connected finite-dimensional K-algebra A can be represented using diagrams. As an example, given the quiver Q depicted below, the diagram below it depicts a module M over KQ. This module has K^6 as its underlying vector space, and it is acted upon by KQ on the right via:

$$(a, b, c, d, e, f)\epsilon_1 = (a, b, c, 0, 0, 0), (a, b, c, d, e, f)\epsilon_2 = (0, 0, 0, d, e, f),$$
$$(a, b, c, d, e, f)\alpha = (d, e, f, 0, 0, 0), (a, b, c, d, e, f)\beta = (0, d, e, 0, 0, 0).$$

$$\begin{tikzcd} 1 & 2 \arrow[l, bend right, "\alpha"'] \arrow[l, bend left, "\beta"] \end{tikzcd}$$

$$\begin{tikzcd} K^3 & K^3 \arrow[l, bend right, "\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}"'] \arrow[l, bend left, "\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}"] \end{tikzcd}$$

References

[1] Ibrahim Assem, Daniel Simson, and Andrzej Skowronski. *Elements of the Representation Theory of Associative Algebras: Volume 1: Techniques of Representation Theory.* Cambridge University Press, 2006.