Visualizing 4-D Polytopes

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Polytopes

- A point set $K \subset \mathbb{R}^d$ is **convex** if for any two points $x, y \in K$ it also contains the straight line segment $[x, y] = \lambda x + (1-\lambda)y : 0 \leq \lambda \leq 1$ between them.

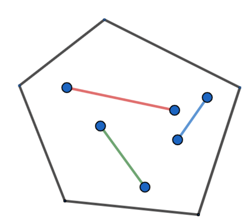

Fig. 1: An example of a convex set in $\mathbb{R}^2$.

- A **polytope** is the generalization of a polygon to any dimension. It has two common descriptions:
  - ⋄ Convex hull of finitely many points:

$$P = conv(x_1, \ldots, x_k) = \left\{\lambda_1 x_1 + \ldots + \lambda_k x_k \,:\, \lambda_i \geq 0, \sum_{i=1}^{k} \lambda_i = 1\right\}.$$

  - ⋄ An intersection of finitely many closed halfspaces in some $\mathbb{R}^d$.

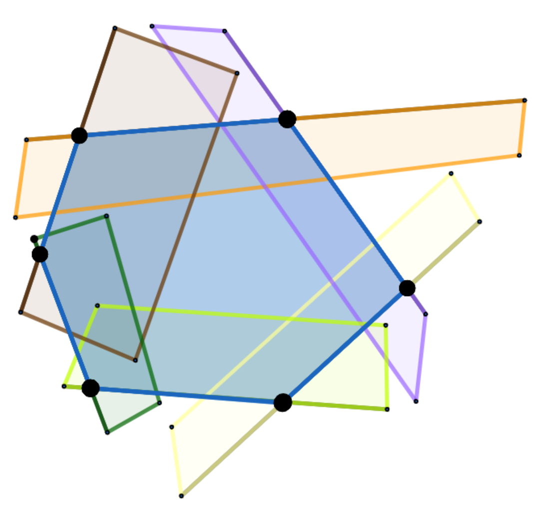

Fig. 2: The halfspace description of a polytope.

- The **dimension** of a polytope is the dimension of the affine hull of its vertices as a vector space.
- The **faces** of a polytope are convex subsets $F \subset P$ such that if $x, y \in P$ and $\lambda x + (1-\lambda)y \in F$ for some $\lambda \in (0, 1)$ then $x, y \in F$.
- The faces of dimensions 0, 1, $dim(P) - 2$, and $dim(P) - 1$ are called vertices, edges, ridges, and facets, respectively.

Poset Structure

The faces of a polytope ordered by inclusion forms a poset. In particular, for any polytope the poset is a lattice called the **face lattice**, denoted $L(P)$.

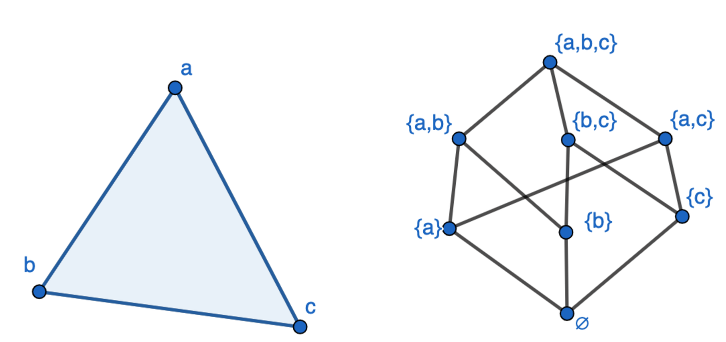

Fig. 3: The face lattice of a triangle.

Schlegel Diagrams

The Schlegel Diagram was created by the German mathematician Victor Schlegel in 1883, to visualize polytopes. The idea is to take a specific projection of the polytope into one lower dimension while retaining important combinatorial information.
Define the map

$$p(\boldsymbol{x}) := \boldsymbol{y}_F + \frac{z - \boldsymbol{a}\boldsymbol{y}_F}{\boldsymbol{a}\boldsymbol{x} - \boldsymbol{a}\boldsymbol{y}_F}(\boldsymbol{x} - \boldsymbol{y}_F),$$

then the *Schlegel Diagram* of P based at the facet F, denoted as $\mathcal{D}(P, F)$, is the image under p of all proper faces of P other than F; that is, it is the set system

$$\mathcal{D}(P, F) := \left\{p(G) : G \in L(P) \setminus \{P, F\}\right\}$$

contained in the hyperplane $H := \text{aff}(F) = \{\boldsymbol{x} \in \mathbb{R}^d : \boldsymbol{a}\boldsymbol{x} = z\}$.

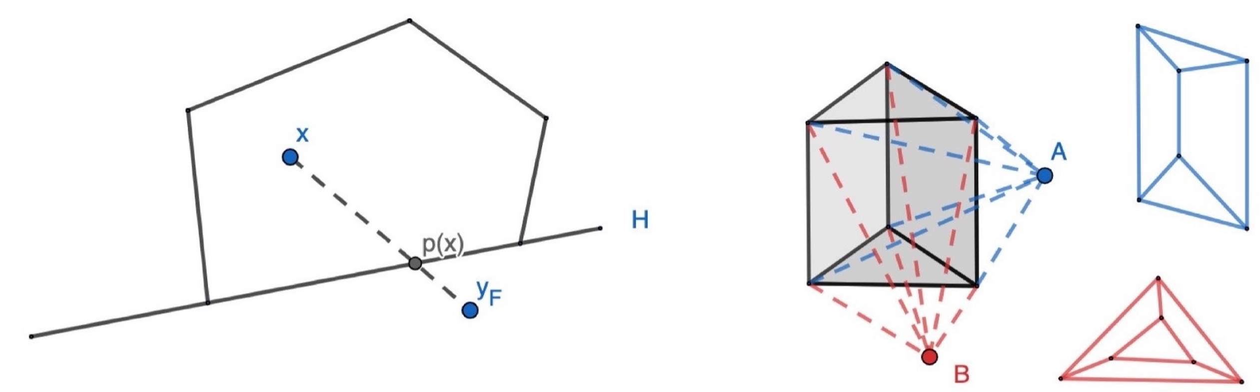

Fig. 4: On the left, visualizing the process. On the right, Schlegel diagrams with two different reference faces.

Schlegel's Theorem

The Schlegel diagram of P based at the facet F is a polytopal subdivision of F that is combinatorially equivalent to the complex $C((\partial P) \setminus F)$ of all proper faces of P other than F.

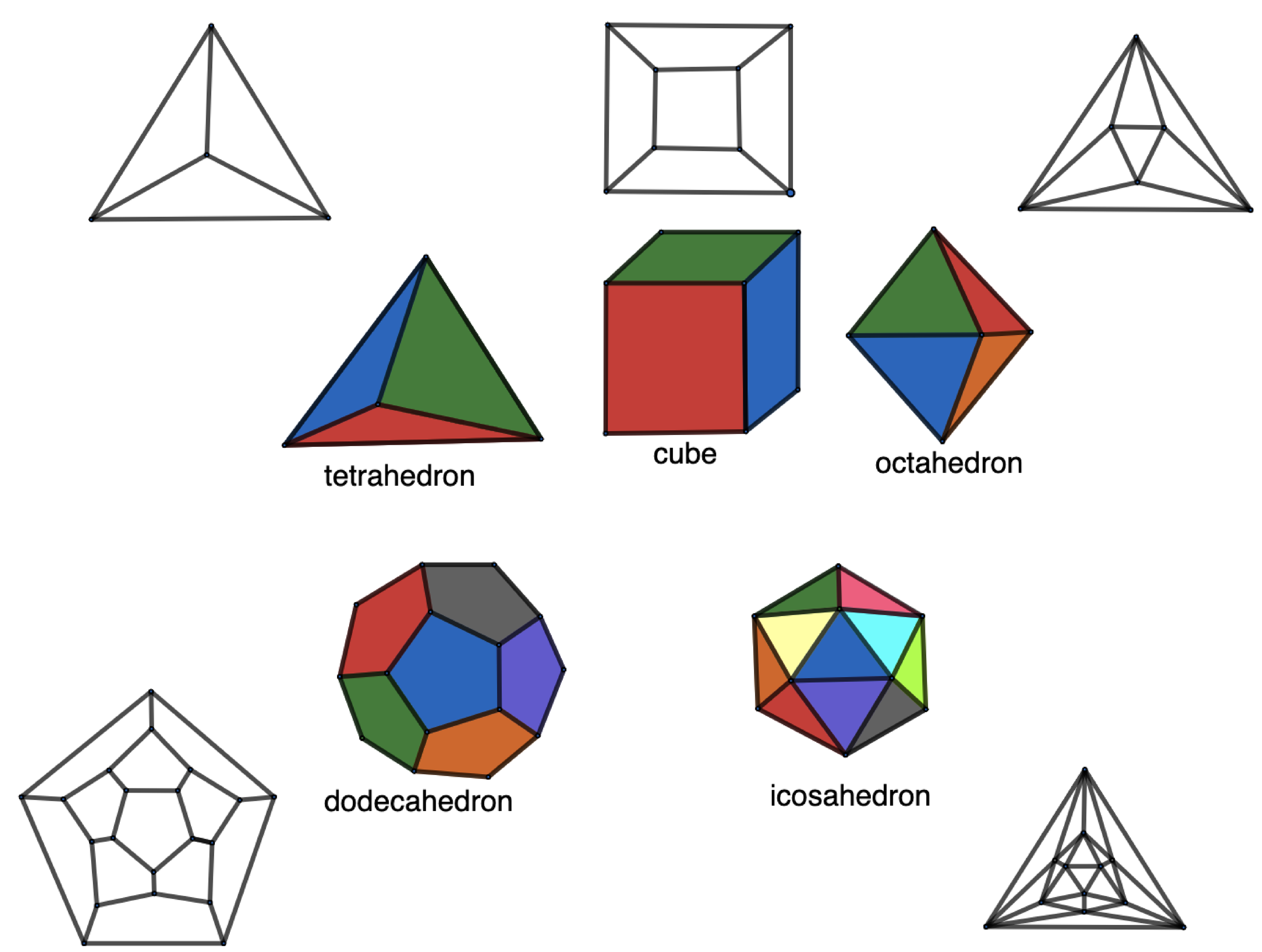

Fig. 5: 5 types of platonic solids and their 2 dimensional Schlegel Diagrams.

4-d Polytopes

- Schlegel diagrams are interesting because they completely encode the combinatorial structure of a d-dimensional polytope into a $(d-1)$-dimensional object, therefore we can use it to visualize a 4-polytope as a 3 dimensional object.

- If a Schlegel diagram of P is given as a polytopal complex $\mathcal{D}$, then we can reconstruct the corresponding facet F of P as $F = |\mathcal{D}|$. In light of Schlegel's Theorem, we can see that the combinatorial isomorphism type of P is determined by the various Schlegel diagrams; thus, we can reconstruct the face lattice of P from it.

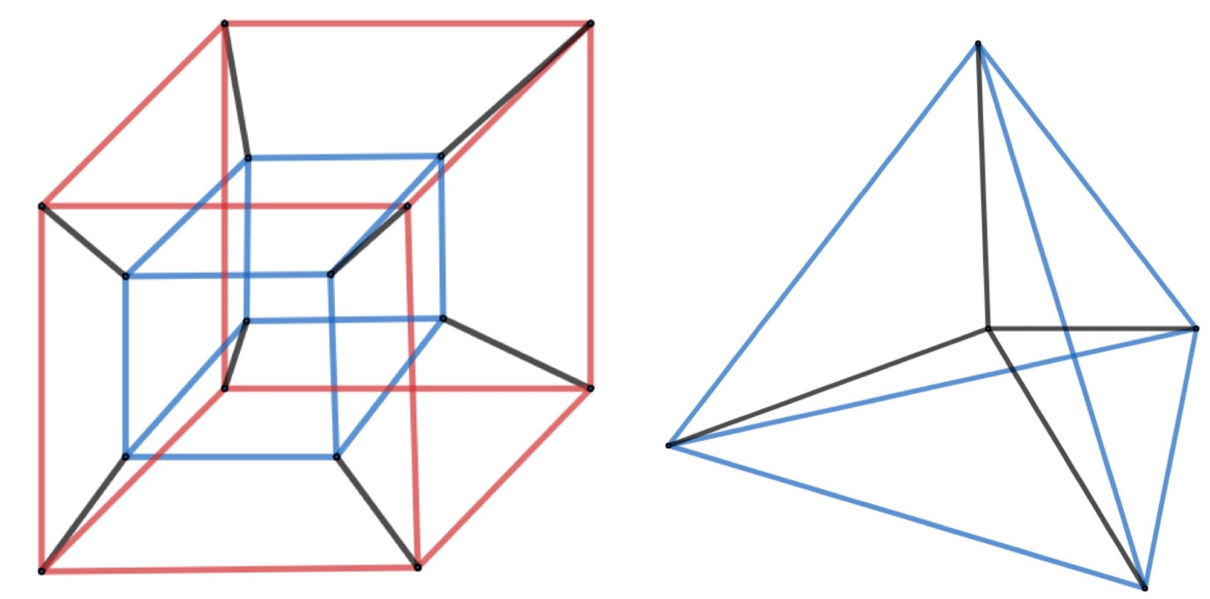

Fig. 6: The Schlegel Diagram for the Hypercube and the Hypersimplex.

Generalizing Schlegel

- Generalizing further we can define the *d-diagram*, which is a polytopal subdivision $\mathcal{D}$ of a d-polytope, $P = |\mathcal{D}| \subseteq \mathbb{R}^d$, such that $G \cap \partial P$ is a face of P for each $G \in \mathcal{D}$.

- Every Schlegel diagram of a d-polytope is a $(d-1)$-diagram.

Reference

[1] Gunter M. Ziegler
Lectures on Polytopes.
Schlegel Diagrams for 4-Polytopes : 127-138, 2007.