The Hyperbolic Plane and Tessellations

Matthew Ro Graduate Mentor: Tom Shifley

Department of Mathematics, University of California, Santa Barbara

Hyperbolic Plane

The **hyperbolic plane** is defined as the metric space consisting of the open half-plane

$$\mathbb{H}^2 = \{(x, y) \in \mathbb{R}^2 | y > 0\} = \{z \in \mathbb{C} | \mathrm{Im}(z) > 0\}$$

This model of the hyperbolic plane is known as the upper half-plane model.
The **hyperbolic length** of a curve γ parameterized by the differentiable vector-valued function

$$t \longmapsto (x(t), y(t)), a \leq t \leq b,$$

as

$$L(\gamma) = \int_a^b \frac{\sqrt{x'(t)^2 + y'(t)^2}}{y(t)} dt$$

This allows us to define a metric on $\mathbb{H}^2$ as the following:
The **hyperbolic distance** between two points $a, b \in \mathbb{H}^2$ is defined as

$$d(a, b) = \inf\{L(\gamma) | \gamma \text{ a path from a to b}\}$$

Geodesics in $\mathbb{H}^2$

A **geodesic** is a locally minimizing curve γ from a to b such that $L(\gamma) = d(a, b)$. In particular, a geodesic is a curve γ such that for every $a \in \gamma$ and for every $b \in \gamma$ sufficiently close to a, the section of γ joining a to b is the shortest curve joining a to b.
Similar to the euclidean plane, geodesics in $\mathbb{H}^2$ can be extended to larger or complete geodesics.
In $\mathbb{R}^2$, we know that geodesics are in the form of straight lines. In $\mathbb{H}^2$, geodesics appear in the form of open semi-circles which meet the real axis at right angles or vertical half-lines which extend from the real line towards ∞.

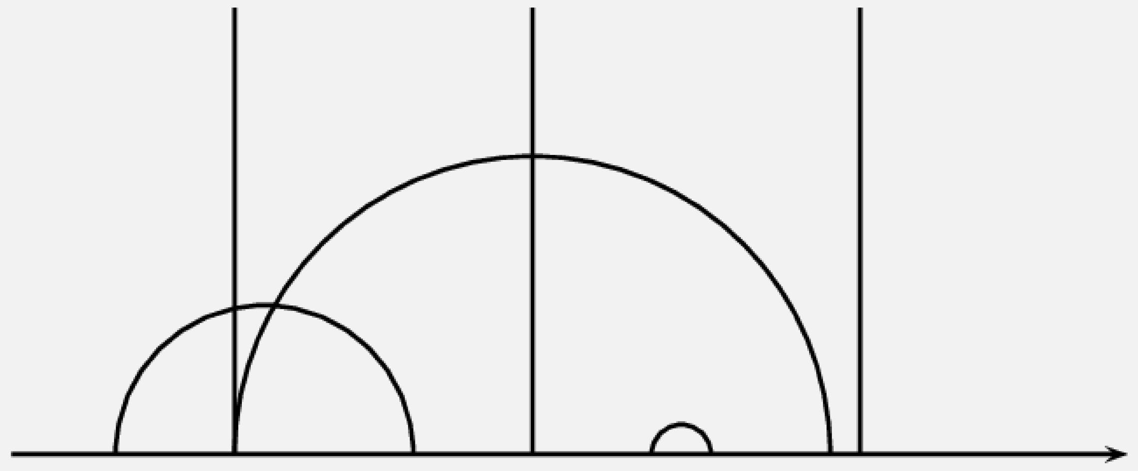

Figure: geodesics in $\mathbb{H}^2$

Isometries of $\mathbb{H}^2$

In $\mathbb{H}^2$ isometries are written in the form of linear fractional maps:

$$f(z) = \frac{az + b}{cz + d}$$

where $a, b, c, d \in \mathbb{R}$ and has the property $ad - bc = 1$
Any isometry of $(\mathbb{H}^2, d)$ are maps in the form of either

$$z \longmapsto \frac{az + b}{cz + d} \qquad z \longmapsto \frac{c\bar{z} + b}{a\bar{z} + d}$$

Orientation preserving Isometries of $\mathbb{H}^2$ are classified in the following cases. Let T be a map then:
Elliptic case: T fixes a unique point in $\mathbb{H}^2$ (similar to a Euclidean rotation)
Parabolic case: T fixes no points in $\mathbb{H}^2$ and one point in $\{\mathbb{R} \cup \infty\}$
Hyperbolic case: T fixes no points in $\mathbb{H}^2$ and fixes two points in $\{\mathbb{R} \cup \infty\}$

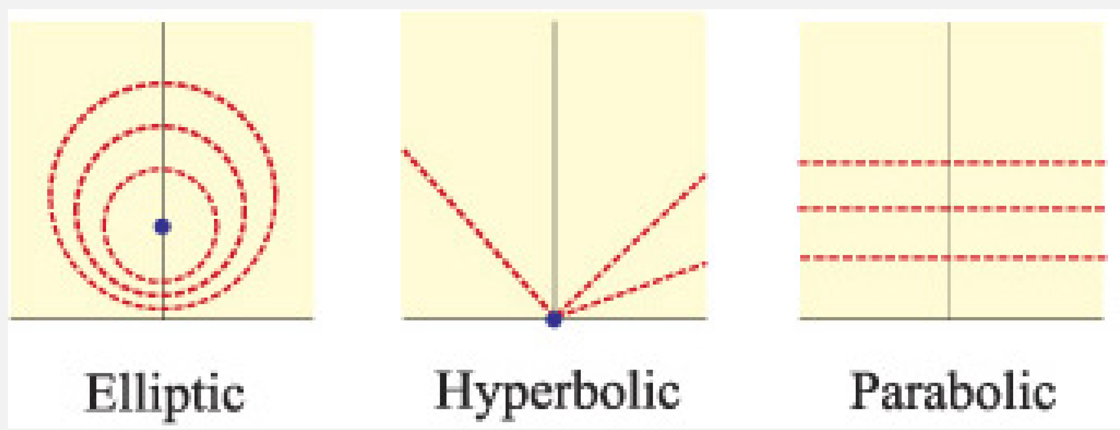

Figure: Orbits of points under the 3 cases

Tessellations by Isometries

A **tessellation** of the Euclidean plane, the hyperbolic plane or the sphere is a family of tiles X_n, $n \in \mathbb{N}$, such that
(1) each tile X_m is a connected polygon in the Euclidean plane, the hyperbolic plane or the sphere;
(2) any two X_m, X_n are isometric;
(3) the X_m cover the whole Euclidean plane, hyperbolic plane or sphere, in the the sense that their union is equal to this space;
(4) the intersection of any two distinct X_m, X_n consists only of vertices and edges of X_m, which are also vertices and edges of X_n;
(5) (Local Finiteness) for every point P in the plane, there exists an ϵ such that the ball of radius ϵ centered at P meets only finitely many tiles X_n.
A **fundamental domain** for the action of a family of bijections Γ on a set X is a connected polygon $\Delta \subset X$ such that as γ ranges over all elements of Γ, the polygons $\gamma(\Delta)$ are all distinct and form a tessellation of X.
If there exists a polygon Δ that is a fundamental domain for a group Γ, then there exists a tessellation Λ with Δ.
Any hyperbolic surface can be realized as $\mathbb{H}^2/\Gamma$ for some discrete subgroup of isometries acting freely on hyperbolic space.

Examples of Hyperbolic Tessellations

We established that there is a correlation between the fundamental domain and tessellating $\mathbb{H}^2$. This allows us to construct beautiful tilings which represent hyperbolic space. The most famous interpretation of these tessellations are drawn by M.C. Escher in his Circle Woodcut series (example below). This tessellation represents hyperbolic geoemtry in the Poincaré disk model in which all the angels and devils are of equivalent size.

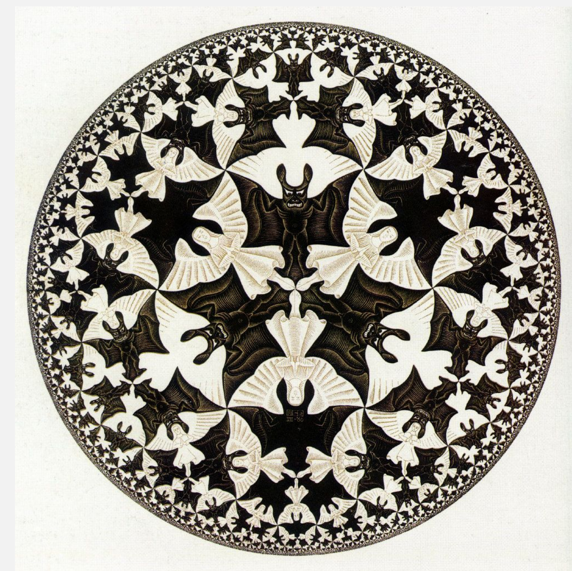

Figure: M.C. Escher Circle Limit IV

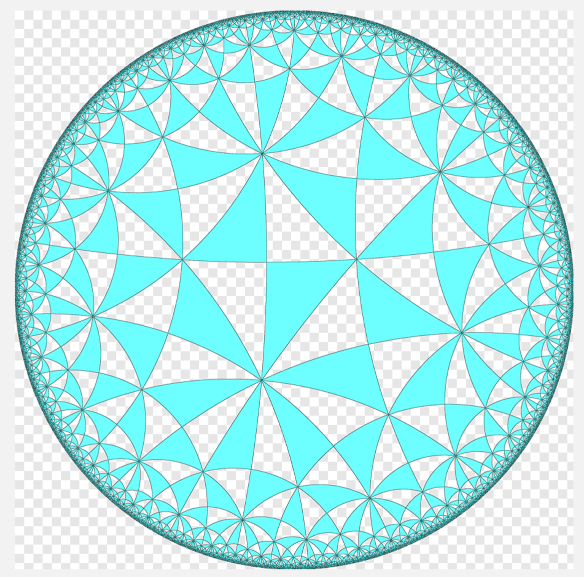

Figure: Tessellation by triangles

References

[1] F. Bonahon.
Low-Dimensional Geometry from Euclidean Surfaces to Hyperbolic Knots.
American Mathematical Society, 1955.

[2] R. Schwartz.
Mostly Surfaces.
American Mathematical Society, 2011.