Connection Theory on Fibre Bundles

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Motivations

The notions of principle fibre bundles and connections help us understand the parallel displacement as well as the covariant differentiation. Since there is no usual way to differentiate a vector field along a curve, we seek for an isomorphism between tangent spaces. The connection on P enables us to specify horizontal subspaces and lift curves on M to horizontal curves on P. Then, the notion of parallel displacement on the bundle provides us with an isomorphism between fibres. In particular, the parallel displacement tells us how to move around a single vector or a frame (contained in the frame bundle).

Background Definitions

Principal Fibre Bundle P(M, G): Let M be a manifold and G a Lie group. A (differentiable) principal fibre bundle $P(M,G)$ consists of a manifold P and an action of G on P satisfying the conditions:

- The right action of G on P is free;
- $M = {}^{P}/_{G}$, and the canonical projection $\pi : P \to M$ is differentiable;
- P is locally trivial ($\pi^{-1}(U) \approx U \times G$).

Fibre: The fibre over x is $\pi^{-1}(x)$. If $u \in \pi^{-1}(x)$, then $\pi^{-1}(x) = \{ua : a \in G\}$ is diffeomorphic to G.

Associated Bundle E(M, F, G, P): Let $P(M,G)$ be a principal fibre bundle and F a manifold on which G acts on the left. Since G acts right on $P \times G$ by $(u, \xi) \mapsto (ua, a^{-1}\xi)$, we obtain a quotient space $E = P \times_G F$. The map $P \times F \to M$ sending (u, ξ) to $\pi(u)$ induces the projection map π_E of E onto M. The differential structure on E comes from the fact that $\pi_E^{-1}(U) \approx U \times F$, induced by $\pi^{-1}(U) \approx U \times G$.

Cross Section: A cross section of E is a map $\sigma : M \to E$ such that $\pi_E \circ \sigma = \mathbb{1}_M$.

Connections in P(M, G)

For each $u \in P$, let $T_u(P)$ denote the tangent space at u. G_u is the subspace consisting of vectors tangent to the fibre through u. A connection Γ in $P(M,G)$ is an assignment of a subspace $Q_u \subseteq T_u(p)$ to each $u \in P$ such that

- $T_u(P) = G_u \bigoplus Q_u$;
- $u \to Q_u$ is invariant by G;
- Q_u depends differentiably on u.

Each vector in $T_u(P)$ splits into two vectors in G_u, Q_u respectively (Vertical & Horizontal).

Geometric Interpretations

By definition of G_u, a vertical vector at $u \in P$ is in the direction of the fibre through u. That is to say, moving in a vertical direction ends up in the same fibre. The geometric reason for defining horizontal directions is to specify a way of moving from fibre to fibre, and with this notion in mind, we can lift vector fields as well as curves from M to P in a horizontal way. This is guaranteed by two propositions below:

Proposition 1: *Given Γ in P and a vector field X on M, there's a unique horizontal lift of X^* invariant by R_a. The converse also holds true.*
(The proof uses the fact: $T_x(M) \cong Q_u$.)

Proposition 2: *Let $\tau = x_t$, $0 \le t \le 1$, be a curve of class C^1 in M. Fix $u_0 \in P$ with $\pi(u_0) = x_0$, there exists a unique lift τ^* starting from u_0.*

The Connection Γ also provides us with a $\mathfrak{g}$-valued 1-form ω, which sends $X \in T_u(P)$ to $A \in \mathfrak{g}$ such that $(A^*)_u$ is the vertical component of X. Since $\omega(X) = 0$ if and only if X is horizontal, we can define the horizontal bundle to be the kernel of ω.

Parallelism and Holonomy Groups

Proposition 2 gives us a way to define the parallel displacement of fibres. Assume that τ has a unique horizontal lift τ^* with end points u_0, u_1. Varying u_0 in the fibre over x_0 ends up with a change in the fibre over x_1. Hence, we obtain a map between fibres, called the parallel displacement along τ. Let $C(x)$ be the loop space at $x \in M$. For each τ, we can view the parallel displacement along τ as an isomorphism from $\pi^{-1}(x)$ to itself. The group structure on $C(x)$ induces a group structure on all such isomorphisms, and we call it the holonomy group of Γ with reference point x. It turns out that the group is isomorphic to a subgroup of G. The concept of holonomy is closely related to the geometric meaning of the curvature, which measures how the parallel transport around closed loops fails to preserve the geometrical data.

Other Important Things and Examples

- **Curvature Form and Structure Equation**
  Since the connection form ω is a pseudotensorial 1-form, the curvature form $\Omega = D\omega = (d\omega)h$ should be a tensorial 2-form. For $X, Y \in T_u(P)$, the structure equation is given by
  $$d\omega(X,Y) = -\frac{1}{2}[w(X), w(Y)] + \Omega(X,Y).$$
  Let $\{e_1, e_2, \cdots, e_r\}$ be a basis for $\mathfrak{g}$ and c^i_{jk}'s be structure constants defined by
  $$[e_j, e_k] = \sum_i c^i_{jk} e_i, \text{ for } j,k = 1, \cdots, r.$$
  If $\omega = \sum_i w^i e_i$ and $\Omega = \sum_i \Omega^i e_i$, then we have the following:
  $$\begin{aligned} d\omega &= -\frac{1}{2}[\sum_i w^i e_i, \sum_i w^i e_i] + \sum_i \Omega^i e_i \\ &= -\frac{1}{2}\sum_{i,j,k} c^i_{jk} w^j \wedge w^k \otimes e_i + \sum_i \Omega^i e_i. \\ d\omega^i &= -\frac{1}{2}\sum_{j,k} c^i_{jk} w^j \wedge w^k + \Omega^i. \end{aligned}$$
  The Bianchi's identity $D\Omega = 0$ is proved by taking exterior differentiation to the structure equation.

- **Example: Connections in a Vector Bundle**
  Let $E(M, \mathbb{R}^m, G, P)$ be a real vector bundle. Given Γ a connection in P and $\tau = x_t$, for each $\xi \in \mathbb{R}^m$, $\tau' = \tau^*\xi$ is a horizontal lift to E provided that τ^* is a horizontal lift to P. Let φ be a cross section of E on τ. The covariant derivative $\nabla_{\dot{x}_t}\varphi$ has the geometric meaning similar to the directional derivative. In particular, it's defined by
  $$\nabla_{\dot{x}_t}\varphi = \lim_{h\to 0}\frac{1}{h}[\tau_t^{t+h}(\varphi(x_{t+h})) - \varphi(x_t)],$$
  where τ_t^{t+h} is the parallel displacement. If ∇ is in the direction of $X \in T_x(M)$, then ∇_X has linearity, functional linearity, and satisfies the Leibniz's law.

- **Example: Linear Connections**
  Let θ be the $\mathbb{R}^n$-valued 1-form on P given by $\theta(X) = u^{-1}(\pi(X))$. If there is a linear connection Γ of M, then we define the torsion form to be $\Theta = D\theta$, which is a tensorial 2-form. The 1-st structure equation is
  $$\begin{aligned} d\theta(X,Y) &= -\frac{1}{2}(w(X)\cdot\theta(Y) - w(Y)\cdot\theta(X)) \\ &+ \Theta(X,Y). \end{aligned}$$

References

[1] Shoshichi Kobayashi and Katsumi Nomizu. *Foundations of differential geometry. Vol. I.* Interscience, 1963.

[2] R.S. Millman and Ann K. Stehney. The geometry of connections. *Am. Math. Monthly*, pages 475–500, 1973.

Future Research

It's also important to understand torsion tensor T and curvature tensor R, defined in terms of covariant differentiation. While T measures whether a connection is good enough (torsion-free), R captures the noncommutativity of the second covariant derivative. These generalize the corresponding concepts in the Riemannian connections, which gives us an opportunity to work on more generalized manifolds.

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