Topological Proof of the Infinitude of Primes

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Abstract

Euclid first presented his proof that there are infinitely many prime numbers well over 2000 years ago. In 1955, mathematician Hillel Fürstenberg published a topological proof of this fact through an investigation of the evenly spaced integer topology. Understanding this proof gave us an opportunity to explore the properties of topological and metric spaces. The goal of our poster is to present Fürstenberg's proof [**1**], provide examples of open and closed sets in Fürstenberg's topology, and explore interesting remarks about this topology.

Furstenberg's Topology

Fürstenberg defines a topology on $\mathbb{Z}$ as follows:

A non-empty subset U of $\mathbb{Z}$ is open iff it is either the empty set, or for each $a \in U$ we can find an $m \geq 1$ such that the arithmetic progression given by $a + m\mathbb{Z}$ is contained in U.

We wish to confirm that Fürstenberg's topology is, indeed, a topology. We do this by verifying the following properties:

1. The empty set $\emptyset$ and the whole set $\mathbb{Z}$ are open.

   The empty set is open by construction of the Fürstenberg topology. For the set $\mathbb{Z}$, let $a = 0$. Then taking $m = 1$, we get the arithmetic progression $0 + 1\mathbb{Z}$. But this is precisely the set $\mathbb{Z}$, which is trivially a subset of $\mathbb{Z}$. Hence $\mathbb{Z}$ is open.

2. An arbitrary union of open sets in the topology is open.

   Let $\{U_i\}$ be an arbitrary collection of open subsets in the topology. Then for each $a \in \bigcup_i U_i$, we have that $a \in U_i$ for some i. Since U_i is open by assumption, then there exists an $m \geq 1$ such that $a + m\mathbb{Z} \subseteq U_i$. Hence, $a + m\mathbb{Z} \subseteq \bigcup_i U_i$, which implies that an arbitrary union of open sets is open.

3. A finite intersection of open sets in the topology is open.

   Let $U_1, ..., U_k$ be open subsets in the topology such that $\bigcap_{i=1}^k U_i \neq \emptyset$. Let $a \in \bigcap_{i=1}^k U_i$. Then for every $i \in \{1, ..., k\}$, there exists some $m_i \in \mathbb{Z}$ such that $a + m_i\mathbb{Z} \subset U_i$. But then $a + m_1 \cdots m_k\mathbb{Z} \subset U_i$ for each i, which means that $a + m_1 \cdots m_k\mathbb{Z} \subset \bigcap_{i=1}^k U_i$. Thus for each element $a \in \bigcap_{i=1}^k$ there exists some $m = m_1 \cdots m_k$ in $\mathbb{Z}$ such that $a + m\mathbb{Z} \subset \bigcap_{i=1}^k$, and hence finite intersections of open sets are open in Fürstenberg's topology.

Because these properties are satisfied, then Fürstenberg's defined topology is indeed a topology.

Infinitude of Primes Proof

Let p be a prime number, and consider the union $\bigcup_p p\mathbb{Z}$ of every integer multiple of a prime number, taken over the set of prime numbers. Note then that because every integer except for 1 and -1 has some prime factor, then $\bigcup_p p\mathbb{Z} = \mathbb{Z} \setminus \{-1, 1\}$.

In a topological space, a set is closed if and only if its complement is open. Note that the set $\{-1, 1\}$ is not open in Fürstenberg's topology, as it is a finite set and hence no arithmetic progression can be contained in it. Thus the complement $\mathbb{Z} \setminus \{-1, 1\}$, or equivalently, $\bigcup_p p\mathbb{Z}$, is not closed.

However, the set $p\mathbb{Z}$, where p belongs to the set of prime numbers, *is* closed. We verify this by showing its complement $\mathbb{Z} \setminus \{p\mathbb{Z}\}$ is open. The set $\mathbb{Z} \setminus \{p\mathbb{Z}\}$ consists of all points $a \in \mathbb{Z}$ such that $p \nmid a$. Thus no integer in the arithmetic progression $a + p\mathbb{Z}$ is divisible by p, and hence $a + p\mathbb{Z} \subset \mathbb{Z} \setminus \{p\mathbb{Z}\}$.

Now suppose for the sake of contradiction that there are only finitely many primes. Then the union $\bigcup_p p\mathbb{Z}$ is a finite union of closed sets. We wish to show that this means $\bigcup_p p\mathbb{Z}$ must be closed, or equivalently, that the complement $\mathbb{Z} \setminus \{\bigcup_p p\mathbb{Z}\}$ is open. Recall that by De Morgan's Law, $\mathbb{Z} \setminus \{\bigcup_p p\mathbb{Z}\}$ is equivalent to $\bigcap_p \mathbb{Z} \setminus \{p\mathbb{Z}\}$. Since we have shown above that $\mathbb{Z} \setminus \{p\mathbb{Z}\}$ is open in this topology, and a finite intersection of open sets is open, then we conclude that $\bigcup_p p\mathbb{Z}$ must be closed.

Finally, since $\bigcup_p p\mathbb{Z}$ is closed if there are finitely many prime numbers, then $\mathbb{Z} \setminus \{-1, 1\}$ must be closed. But we showed above that this is not the case, and have therefore reached a contradiction. We conclude that there must be infinitely many prime numbers.

Example

Arithmetic progressions are both open and closed.

Let $a + m\mathbb{Z}$ be any arithmetic progression. Consider $a + mb \in a + m\mathbb{Z}$. Then, taking the same m, we have $(a + mb) + m\mathbb{Z} = a + m(\mathbb{Z})$. Then clearly $a + m(\mathbb{Z}) \subseteq a + m(\mathbb{Z})$, and we conclude that any arithmetic progression is open. To show it is also closed, we consider its compliment. The compliment of an arithmetic progression is the finite union of arithmetic progressions of the form, $r + m\mathbb{Z}$ for $0 \leq r \leq m - 1$ and $r \not\equiv a(\mod m)$. Note that any $r + m\mathbb{Z}$ is open because it is an arithmetic progression. Further, since these are subsets of the topology, their union is also open. Thus any arithmetic progression is also closed.

Remarks

Remark 1: We showed in our example that arithmetic progressions are both open and closed. Thus this topological space is disconnected. The paper [**2**] shows that this topological space $(\mathbb{Z}, \mathcal{T})$ is totally disconnected.

Remark 2: The Fürstenberg topology is metrizable. [**2**]

For $n \in \mathbb{Z} \setminus \{0\}$, we define the norm of n as,

$$||n|| := \frac{1}{max\{k \in \mathbb{N} : 1|n, ..., k|n\}}.$$

In other words, the norm of n is the reciprocal of the largest $k \in \mathbb{N}$ such that the natural numbers $1, ..., k$ are divisors of n. Further, let $||0|| := 0$. Then, we define a metric on $\mathbb{Z}$ as,

$$d(m, n) := ||m - n||.$$

It can be proven that $A \in \mathcal{T}$ if and only if for every $a \in A$, there exists a $b \in \mathbb{Z}$ such that $a + b\mathbb{Z} \subset A$, and if we let $r = \frac{1}{b}$, then $B(a, r) \subset A$. Thus this metric induces our desired topology.

Remark 3: This metric space is not complete.

It is sufficient to show that this is not a Baire space. A space X is Baire if given any countable collection $\{A_n\}$ of closed sets of X with empty interiors, the union of these sets has an empty interior [**3**]. In this topology finite sets are not open, and hence sets of single points are not open. In addition, since single points are closed in a metric space, then their interior must be the empty set. But then a countable collection of such sets creates an arithmetic progression, which is open in our topology, and hence does not have an empty interior. Thus our metric space is not Baire which implies not complete [**3**].

Acknowledgements

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References

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[2] L. Lovas, I. Mezo, *On an exotic topology of the integers*, (2010)

[3] J. R. Monkres,*Topology 2nd edition*, (2015)