Useful Ideas in Analytic Number Theory

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The results presented here are taken from...

This poster shows a few results that we learned from the book *Introduction to Analytic Number Theory* by Tom Apostol.

Convolutions

In this section we establish the notion of Dirichlet Convolution and we give some basic results. Let $\mathcal{F}$ be the set of arithmetic functions with $f(1) \neq 0$. Define $*$ on $\mathcal{F}$ to be such that $f * g(n) = \sum_{d|n} f(d)g(\frac{n}{d})$. Our first result is that $(\mathcal{F}, *)$ is an abelian group with the identity $I(n) = \begin{cases} 1 & \text{n = 1} \\ 0 & \text{otherwise} \end{cases}$

From here, the Mobius Inversion formula immediately follows. Namely, for $f, g \in \mathcal{F}$, we have that if $f(n) = \sum_{d|n} g(d)$, then

$$g(n) = \sum_{d|n} f(d)\mu(\frac{n}{d})$$

with the converse also true. The proof is immediate since we begin with $f = g*\mu$. Then since μ is associative under $*$, it follows that $(g*\mu)*\mu = g*(\mu*\mu) = g*I = g$. The reverse direction is proven similarly.

Relation Between Euler's Phi Function and Mobius Function

Mobius inversion gives several important results. We illustrate one such result with the Euler Phi function. For a natural number n, we define

$$\phi(n) = \sum_{k=1}^{n} I_k$$

where $I_k = 1$ if $(k, n) = 1$ and 0 otherwise. First, we show that

$$\sum_{d|n} \phi(d) = n$$

. Define the set $S = \{1, 2, ..., n\}$. Then for each divisor d of n, take the set

$$A(d) = \{k \mid (k, n) = d \quad \text{and} \quad k \in S$$

. Then we see that all such $A(d)$ are disjoint and

$$\bigcup_{d|n} A(d) = S.$$

Now for each set $A(d)$, we have $k \in A(d)$ only if $(k, n) = d$ and $1 \leq k \leq n$. These last two conditions are equivalent to $(\frac{k}{d}, \frac{n}{d}) = 1$ and $\frac{k}{d} \leq \frac{n}{d}$. Therefore, we conclude that $|A(d)| = \phi(\frac{n}{d})$ and so

$$\sum_{d|n} \phi(d) = \sum_{d|n} |A(d)| = \sum_{d|n} \phi(\frac{n}{d}) = n = \sum_{d|n} \phi(d)$$

which is the desired result.

Now we may immediately apply Mobius inversion to see that

$$\phi = \sum_{d|n} \mu(d)\frac{n}{d}$$

A New Way of Summation

This section uncovers a striking reinterpretation of a double summation as the number of lattice points beneath a graph. While seemingly innocuous, this method is far-reaching and can be used to derive several identities and estimates which prove useful in analytic number theory. Here, it is used to derive Dirichlet's asymptotic formula for the partial sums of the divisor function $d(n)$, defined by

$$d(n) = \sum_{d|n} 1$$

The statement is the following: for all $x \geq 1$ we have

$$\sum_{n \leq x} d(n) = x \log x + (2\gamma - 1)x + O(\sqrt{x})$$

where γ is the Euler-Mascheroni constant.
Proof: From the above definition we have that

$$\sum_{n \leq x} d(n) = \sum_{n \leq x} \sum_{d|n} 1$$

This is a double sum extended over n and d. Since $d|n$ we can write $n = qd$, and extend the sum over all pairs of positive integers q, d with $qd \leq x$. Thus the sum can be rewritten as

$$\sum_{qd \leq x} 1$$

We now use the remarkable idea of interpreting this sum as being extended over certain lattice points in the qd-plane. Namely, this sum counts the total number of lattice points in the upper right quadrant underneath the graph $qd = x$. Equivalently, it is the number of lattice points (q, d) in the upper right quadrant lying on the curves $qd = n$ for $n = 1, 2, \ldots, \lfloor x \rfloor$ as shown in the figure below.

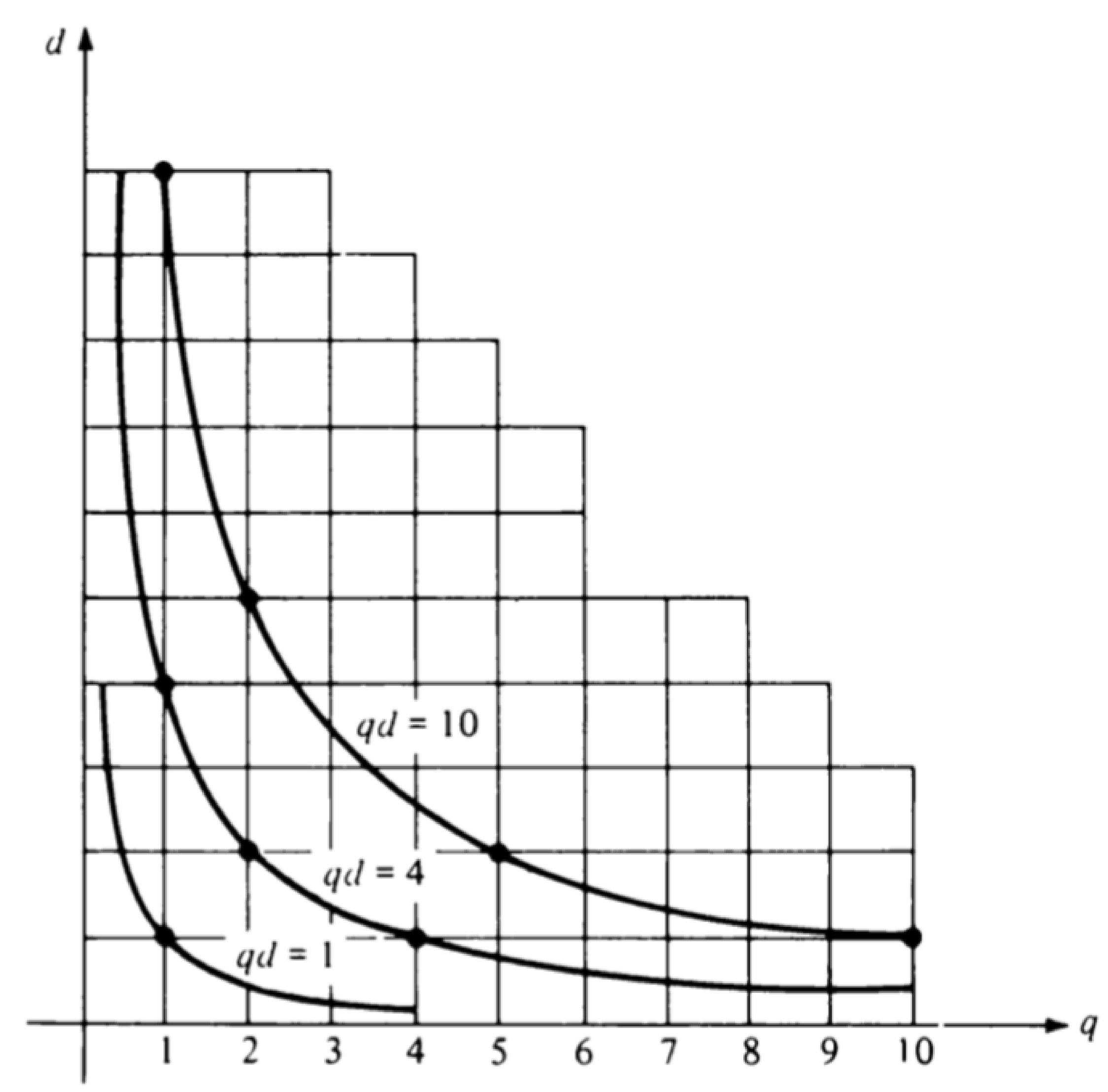

Fig. 1: Big fancy graphic.

To count the number of such lattice points, take advantage of the symmetry of this hyperbolic region about the line $q = d$. The total number of lattice points in this region is equal to twice the number below the line plus the number of lattice points on the bisecting line segment.

A New Way of Summation (cont.)

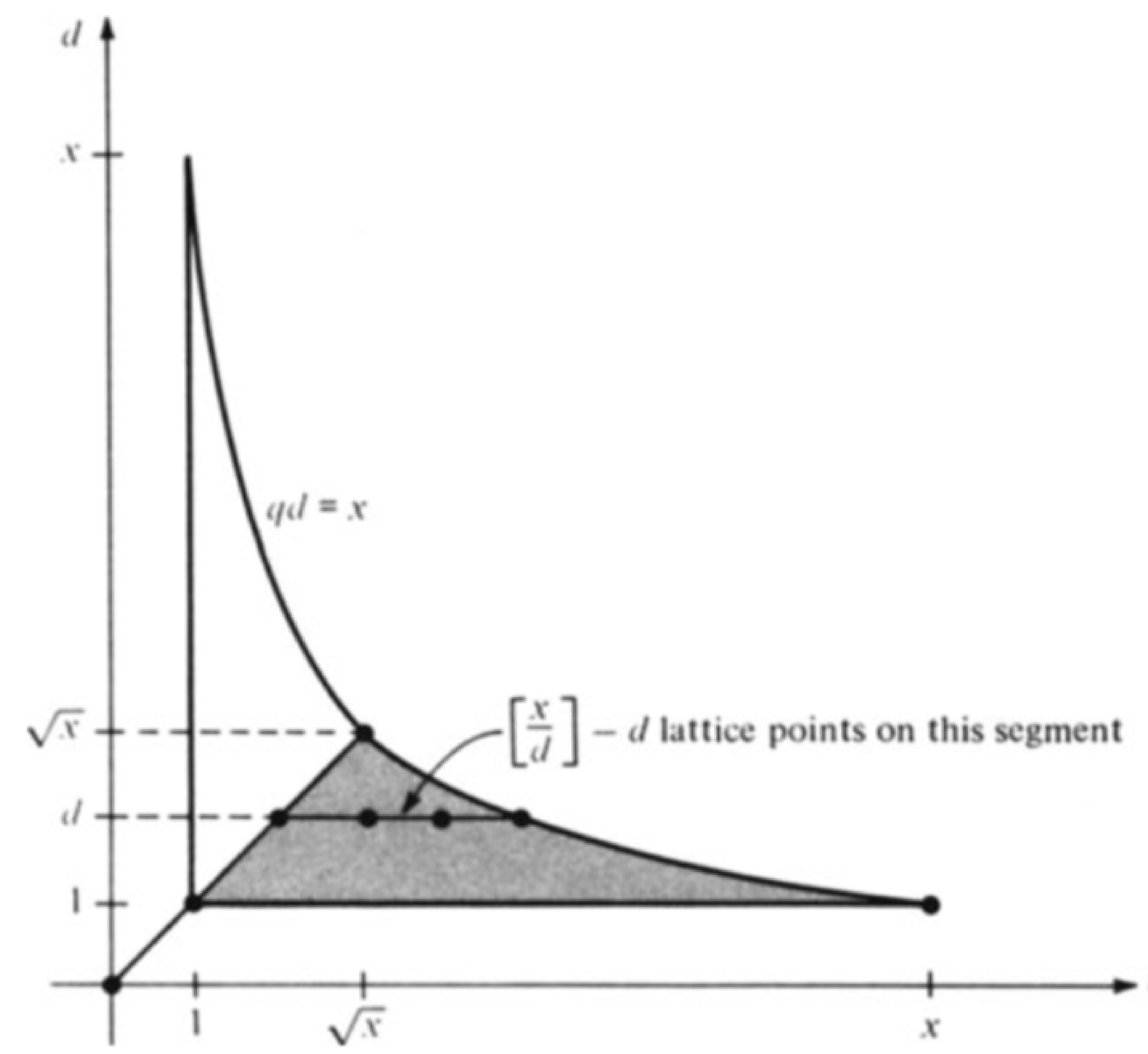

Fig. 2: Big fancy graphic.

From the figure above one can see that this is equal to

$$\sum_{n \leq x} d(n) = 2 \sum_{d \leq \sqrt{x}} \left(\left\lfloor \frac{x}{d} \right\rfloor - d \right) + \lfloor \sqrt{x} \rfloor$$

Now we use the relation $\lfloor y \rfloor = y + O(1)$ and the following formulas

$$\sum_{n \leq x} \frac{1}{n} = \log n + \gamma + O\left(\frac{1}{n}\right)$$

$$\sum_{n \leq x} n^{\alpha} = \frac{x^{\alpha+1}}{\alpha + 1} + O(x^{\alpha}), \quad \alpha \geq 0$$

to obtain

$$\begin{aligned} \sum_{n \leq x} d(n) &= 2 \sum_{d \leq \sqrt{x}} \left(\frac{x}{d} - d + O(1) \right) + O(\sqrt{x}) \\ &= 2x \sum_{d \leq \sqrt{x}} \frac{1}{d} - 2 \sum_{d \leq \sqrt{x}} d + O(\sqrt{x}) \\ &= 2x \left\{ \log \sqrt{x} + \gamma + O\left(\frac{1}{\sqrt{x}}\right) \right\} - 2 \left\{ \frac{x}{2} + O(\sqrt{x}) \right\} + O(\sqrt{x}) \\ &= x \log x + (2\gamma - 1)x + O(\sqrt{x}) \end{aligned}$$

as desired.

Acknowledgements

Thanks to the DRP for presenting us with the opportunity to learn analytic number theory, a subject we don't see in our usual classes. Thanks to Kathy for going through Apostol's book with us and leading us along the way.