Braids in Knot Theory

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What is Knot Theory?

A **knot** is essentially a piece of string that was knotted with itself and had its ends reconnected. A significant part of knot theory is figuring out which knots are distinct from each other and the trivial knot (basically a circle). Braids can help with describing knots and differentiating between them.

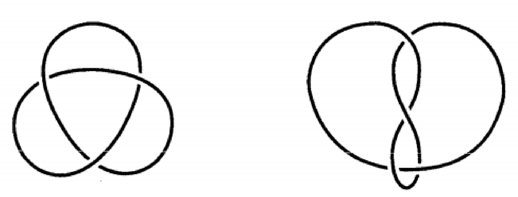

Figure 1:Two simple knots

What is a braid?

A **braid** is a set of finitely many strings that are attached to two horizontal bars. The strings always travel strictly downwards, meaning they can curve but cannot intersect any horizontal plane more than once. Braids are equivalent if you can rearrange the strings while keeping them fixed to the two horizontal bars (which are infinitely long).

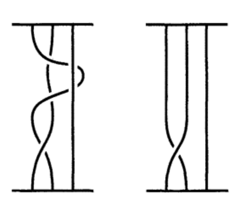

Figure 2:Two equivalent braids

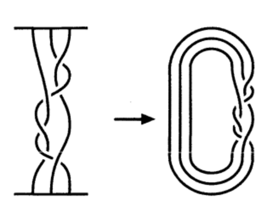

Figure 3:Closure of a braid

If we join the two horizontal bars such that the ends of the strings meet up, then we will have a knot. This is called the closure of the braid or a closed braid representation of a knot. Every knot is a closed braid! This was proven by J.W. Alexander, an influential researcher in Knot Theory, in 1923.

How do we categorize braids?

The simplest way to categorize braids is to look at the number of strings that it has. The **braid index** is the least number of strings needed in a closed braid representation of a knot. This doesn't change no matter how we draw the knot or how many strings we start with.

In order to describe a given braid, we will look at the intersections of the braid and what order they occur in. Take two consecutive strings n and $n+1$. If string n crosses over string $n+1$, then it is called a σ_n crossing. If string $n+1$ crosses over string n, then it is a σ_n^{-1} crossing.

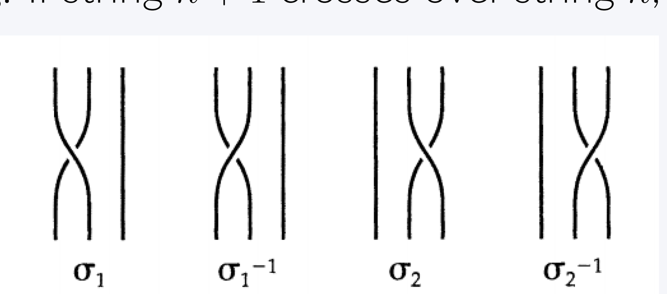

Figure 4:Examples of crossings

We can always tweak the braid a little such that no two crossings occur at the same height. Then starting from the top horizontal bar, we can travel down the braid to the bottom horizontal bar, categorizing the intersections as we go. This will give us a sequence $\sigma_i^{\pm 1}\sigma_j^{\pm 1}...\sigma_k^{\pm 1}$ called the **word** of the braid.

Markov's Theorem

Markov's Theorem states that two knots are Markov equivalent if their closed braid representations are related through a sequence of five operations.

1. **Rule 1:** In a word, σ_i^{-1} cancels σ_i and σ_i cancels σ_i^{-1}. Whenever σ_i^{-1} and σ_i are next to each other in a word, that part of the sequence can deleted. In a braid, that combination of intersections cancels each other out.

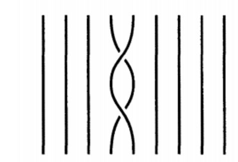

Figure 5:σ_4^{-1} cancels σ_4

2. **Rule 2:** In a word, $\sigma_i\sigma_{i+1}\sigma_i$ is the same as $\sigma_{i+1}\sigma_i\sigma_{i+1}$. Visually what happens is that keeping string i and string $i+2$ fixed, we can slide string $i+1$ between them to get either word.

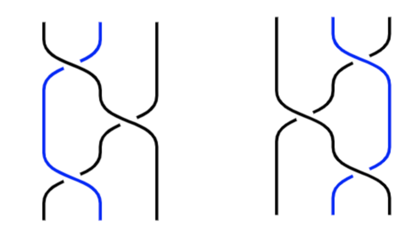

Figure 6:The intersections can be rearranged

3. **Rule 3:** When $|i-j|>1$, we can switch σ_i and σ_j. We can do this because the intersections won't be next to each other. They will be on different strings, so we can switch their relative position without affecting the braid.

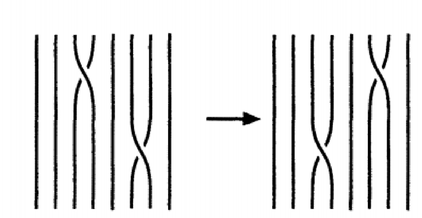

Figure 7: σ_3 and σ_6 are switched

4. **Conjugation:** We can multiply the end of the word by σ_i and the beginning by σ_i^{-1}. This will add a twist at the top and bottom of the braid that are in opposite directions. When taking the closure of the braid, these twists cancel each other out.

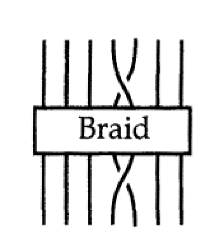

Figure 8:Two twists were added

5. **Stabilization:** For any braid, we can add another string by adjoining $\sigma_n^{\pm 1}$ to a word for a n string braid. If a word of a n string braid that ends with $\sigma_n^{\pm 1}$, we can delete it without altering the braid. This allows us to add or delete loops in a closed braid.

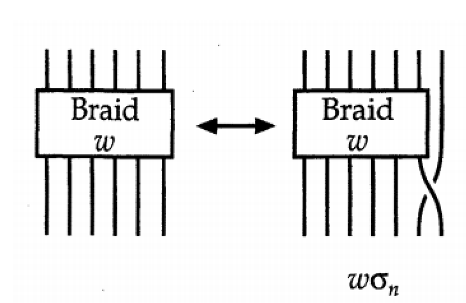

Figure 9:A loop was added

Group of n-string Braids

The Operation: Multiplication

The operation for multiplying braids is essentially sticking them on top of each other. For each braid the endpoints on the top match up with the endpoints on the bottom. Given two braids with n strings each, we can take the bottom endpoints for the first braid and superpose them over the top endpoints for the second braid. Then we will have one braid with n strings that is the product of the two. The words of the two braids are composed end to end as well.

This is a group!

By multiplying n-string braids, we get a group. A group is a set that is closed under the operation, has an identity, has inverses, and is associative.

Identity

The identity is just the braid with n strings such that each string goes directly to the opposite endpoint. So the word is "blank" essentially. Thus when we multiply by it, the word of the other braid is not affected.So we will always get the other braid as the product.

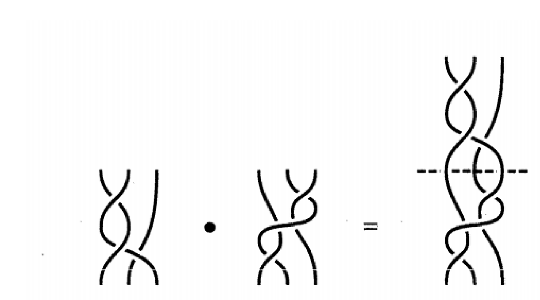

Figure 10:Multiplying two braids

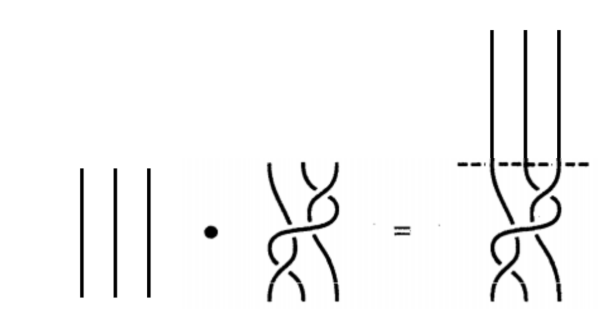

Figure 11:The identity preserves the braid

Inverse

To get the inverse of any braid, that the elements of the word and write them in reverse order and with each element raised to -1. Then when you multiply the two braids, the words will cancel each other out. This is the left and right inverse.

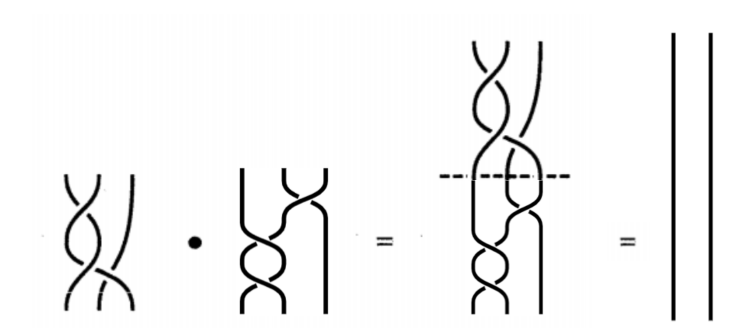

Figure 12:The inverse returns the identity

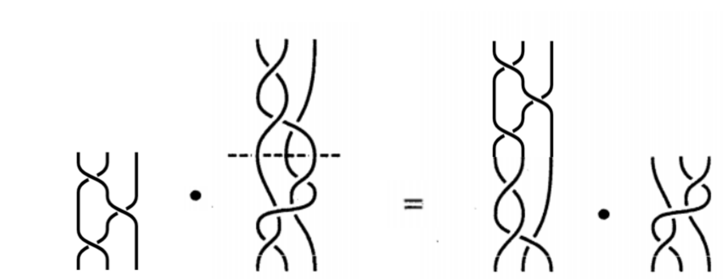

Figure 13:Associativity ensures the result is the same

Associativity

Associativity means that a(bc) = (ab)c for any three braids a,b,c. In either case, we connect the bottom endpoints of a to the top endpoints of b, and the bottom endpoints of b to the top endpoints of c. This occurs regardless of whether we multiply ab or bc first.

References

[1] Adams Colin C. The Knot Book. *American Mathematical Society*, 2004.