Topology and Applications

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What is Topology?

Topology is the study of objects in spaces that incur transformations such as deformation, stretching, and bending. Two objects are homeomorphic if they can be deformed or manipulated in such a way such that they can be transformed into each other. This is done by a continuous mapping, where a continuous function maps between two topological spaces. As an example, the letter L is homeomorphic to W since they both can be deformed into a straight line.

What is a Homotopy?

Homotopy is the idea of continuously deforming a path while keeping its endpoints fixed.

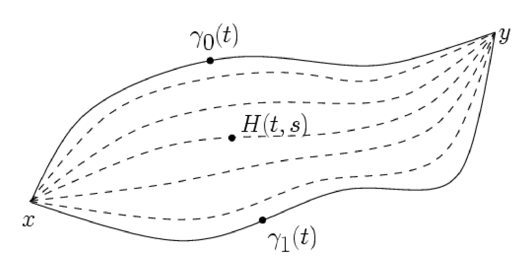

Figure 1: Example of homotopic curves with endpoints x and y

Formally, a homotopy of paths in a topological space X is defined as a family of functions $f_t : I \mapsto X$ with $0 \leq t \leq 1$ such that:

a. the endpoints of the path $f_t(0) = x_0$ and $f_t(1) = x_1$ are independent of t
b. the function $H : I \times I \mapsto X$, defined by $H(s,t) = f_s(t)$, is continuous.

(With respect with Figure 1 above, we would say that γ_0 and γ_1 are **homotopic** and write $\gamma_0 \simeq \gamma_1$)

In fact, the relation of homotopy on paths with fixed endpoints in any topological space is an equivalence relation! This means that we can group the paths that have the same fixed endpoints together since they are equivalent. Note that we call the equivalence classes of a path f under the equivalence relation of homotopy the **homotopy classes of** f.

The Fundamental Group

Note that paths with the same starting and ending point x_0 (called the **basepoint**) are called **loops**.

We say that the set of all homotopy classes $[f]$ of loops $f : I \mapsto X$ at the basepoint x_0 is denoted by $\Pi_1(X, x_0)$, where $\Pi_1(X, x_0)$ is called the **fundamental group**.

In other words, we define the **fundamental group** of a space X so that its elements are loops in X starting and ending at a fixed basepoint $x_0 \in X$. But, two such loops are regarded as determining the same element of the fundamental group if one loop can be continuously deformed to the other within the space X.

- The reason why we care about the fundamental group of a topological space is because it tells us information about the basic shape of or the amount of holes in that space. For instance, when considering the circle $S^1 = \{ (x,y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1 \}$, we notice that the each homotopy class consists of all loops wrapping around the circle a certain number of times and the product of a loop that wraps around m times and another loop that wraps n times around is simply a single loop that winds around $m+n$ times. Thus, the fundamental group of a circle $\Pi_1(S^1, x_0) = \mathbb{Z}$ and is isomorphic to the additive group of integers $(\mathbb{Z}, +)$. The fundamental group of a circle tells us that it has one generator going around the circle; in other words, this space has one hole (as you can see in Figure 2 below).

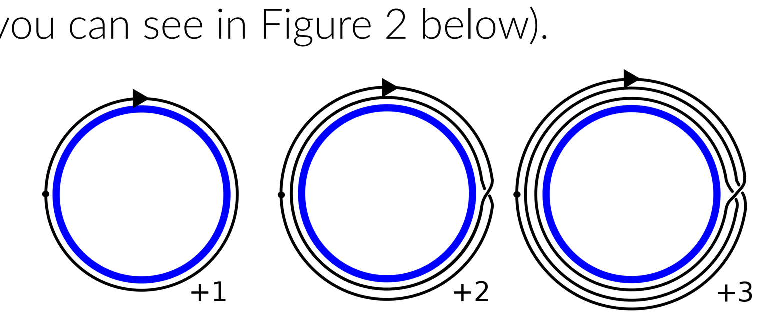

Figure 2: Elements of the homotopy group of the circle

Metric Space Terminology

Let X be a non-empty set and d be a real-valued function on $X \times X$ such that for $a, b, c \in X$:

- $d(a,b) \geq 0$, and $d(a,b) = 0$ if and only if $a = b$,
- $d(a,b) = d(b,a)$, and
- $d(a,c) \leq d(a,b) + d(b,c)$

Then, d is said to be the **metric** on X, the pair (X,d) is called a **metric space**, and $d(a,b)$ is referred to as the **distance** between a and b.

Metric Spaces in Linear Algebra

Let V be a vector space over the field,F, of either $\mathbb{R}$ or $\mathbb{C}$. A norm $||\ ||$ on V is a map: $V \to \mathbb{R}$ such that for all $a, b \in V$ and $\lambda \in \mathbb{F}$,

- $||a|| \geq 0$, and $||a|| = 0$ if and only if $a = 0$,
- $||a + b|| \leq ||a|| + ||b||$, and
- $||\lambda a|| = |\lambda| ||a||$.

A **normed vector space** $(V, ||\ ||)$ is a vector space V with a norm $||\ ||$.

We let $(V, ||\ ||)$ be any normed vector space. Then, there is a corresponding metric, d, on the set V given by

$$d(a,b) = ||a - b||, \text{ for } a, b \in V$$

We see that $\mathbb{R}^n$ is a normed vector space if we let

$$|| < x_1, x_2, ..., x_n > || = \sqrt{x_1^2 + x_2^2 + ... + x_n^2} \text{ for } x_1, x_2, ..., x_n \in \mathbb{R}$$

So, $\mathbb{R}^n$ becomes a metric space if we put

$$d(< a_1, a_2, ..., a_n >, < b_1, b_2, ..., b_n >) = || < a_1 - b_1, a_2 - b_2, ..., a_n - b_n > ||$$
$$= \sqrt{(a_1 - b_1)^2 + (a_2 - b_2)^2 + ... + (a_n - b_n)^2}.$$

Thus, the family of normed vector spaces are examples of metric spaces.

Metric Space Terminology Continued

In a normed vector space $(V, ||\ ||)$ the **open ball** with center a and radius r is defined to be the set

$$B_r(a) = \{x : x \in N \text{ and } ||x - a|| < r\}$$

We let (X, d) be a metric space and r any positive real number. Then, the open ball about $a \in X$ of radius r is the set

$$B_r(a) = \{x : x \in N \text{ and } d(a,x) < r\}$$

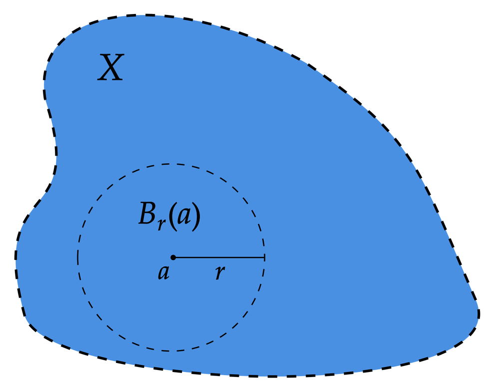

Figure 3: Visualization of an open ball in an open set X

Graphs

A **graph** G is a construction $G = (V, E)$, where V is the set of vertices and E is a set of edges. We use topology to understand graphs and their relationship to a given plane. **Planarity** refers to the property of G such that it can be drawn with no edge crossings.

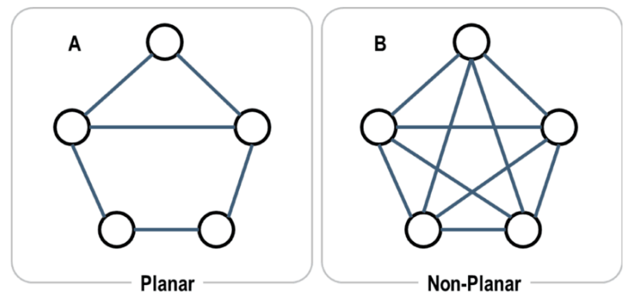

Figure 4:Example of a Planar and a Non-Planar Graph

Torus

A **torus** refers to a topological space that is homeomorphic to the Cartesian product of two circles: $\mathbb{T} = \mathbb{S}^1 \text{x} \mathbb{S}^1$. A torus is constructed by taking a rectangle and connecting opposite sides to each other. Tori are useful in that they help identify many properties of a graph such as its planarity on the tori and its crossing number. The **crossing number** of G, denoted by $cr(G)$, is the lowest number of edge crossings of a plane drawing of the graph G. Crossing numbers are useful because it helps identify efficient pathways (edges) between locations (nodes).

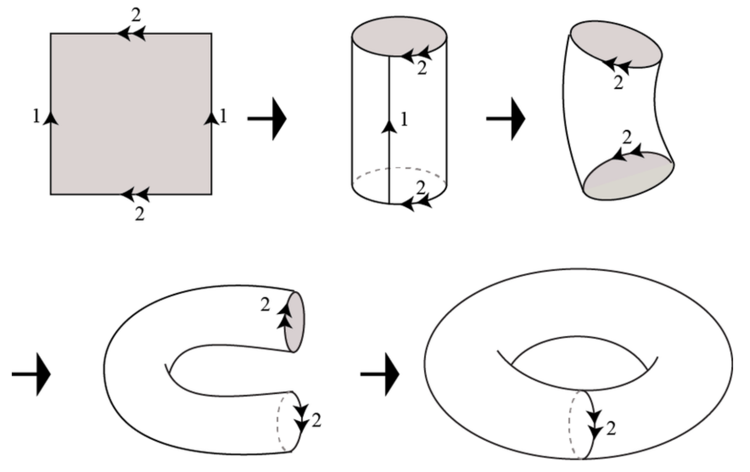

Figure 5: Torus Construction

Genus

Suppose we have a nonplanar graph with 1 unavoidable intersection, meaning $cr(G) = 1$. If we add a handle to a sphere and relieve the unavoidable intersection, we get a torus. If we have graph with k unavoidable intersections, we can add k handles and create a k-hole torus. The **genus** of a graph is defined as minimal integer k of handles added to a sphere such that the graph G can be drawn without crossing itself. In doing so, we see that it is possible to achieve planarity on a k-hole torus. Below are examples of different surfaces of genus $0, 1, 2,$ and 3 respectively.

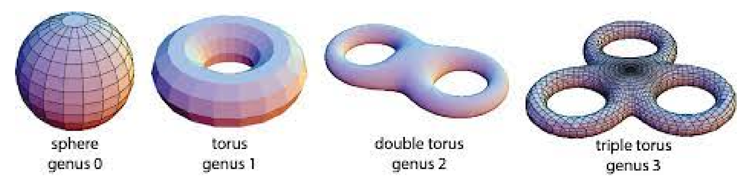

Figure 6: Tori of different genus

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