Classifying Lie Algebras Using Cartan's Criteria

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Introduction

In the elementary study of Lie algebras there are two main characteristics that we look at in terms of classifying Lie algebras; simplicity and solvability. Simplicity also comes with the study of semisimplicity. These two concepts help us to understand the characteristics and structures of Lie algebras, sub-Lie algebras, and ideals of Lie algebras. Thus understanding when Lie algebras are solvable or simple is very helpful. The Killing form and Cartan's Criteria give us methods to assess if Lie algebras are simple, solvable, and what it means to be semisimple. This poster is designed to introduce these methods and provide examples of their applications to common Lie algebras such as $\mathrm{sl}(2,\mathbb{C})$, the Lie Algebra consisting of all 2x2 matrices with trace 0 and $\mathrm{gl}(2,\mathbb{C})$, the Lie algebra consisting of all 2x2 matrices.

Definitions

- **Lie Algebra**: A **Lie Algebra** over a field, F, is an $F-$vector space L, with a bilinear map

$$L * L \to L,\ (x,y) \mapsto [x,y]$$

This bilinear map [,] is called the **Lie Bracket**, and it satisfies the following properties:

$$[x,x]=0,\ \forall x \in L$$

$$[x,[y,z]]+[y,[z,x]]+[z,[x,y]]=0,\ \forall x,y,z \in L$$

For the purposes of this poster we will use $[x,y]=xy-yx$

- **Solvable**: The Lie Algebra L is **solvable** if for some $m \geq 1$ we have $L^{(m)}=0$, where $L^{(1)}=[L,L]$ and $L^{(m)}=[L^{(m-1)},L^{(m-1)}]$, for $m \geq 2$
- **Radical**: The largest solvable ideal is the **radical** of L, or **rad(L)**
- **Simple**: A non abelian Lie algebra, L, is **simple** if it has no other ideals besides 0 and L itself
- **Semisimple**: A non zero, finite dimensional Lie algebra, L, is **semisimple** if rad(L)=0
- **Adjoint Representation**: The **adjoint representation** is a Lie homomorphism defined as

$$\mathrm{ad}: L \to \mathrm{gl}(L),\text{ where } (\mathrm{ad}x)(y) := [x,y]$$

- **Non-degenerate**: A bilinear form β is **non-degenerate** if $V^{\perp} := \{x \in V \mid \beta(x,y)=0,\ \forall y \in V\}=0$, where V is a vector space.

The Killing Form

The **Killing Form** is a symmetric bilinear form, which will be used in Cartan's criteria as a tool to help us assess solvability and semisimplicity of Lie Algebras. It is defined as follows:

$$\kappa(x,y) := \mathrm{tr}(\mathrm{ad}\ x \circ \mathrm{ad}\ y) \text{ for } x,y \in L, \text{a complex Lie algebra}$$

Notice that the symmetric property of the Killing form is obvious, since $\mathrm{tr}(AB)=\mathrm{tr}(BA)$. Let's further our understanding by looking at $\mathrm{sl}(2,\mathbb{C})$, which has the basis:

$$e := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, f := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, h := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

The Killing Form Cont.

First we compute $\mathrm{ad}\,e$, $\mathrm{ad}\,f$, $\mathrm{ad}\,h$, using the formula $\mathrm{ad}\,e := ((\mathrm{ad}\,e)(e), (\mathrm{ad}\,e)(f), (\mathrm{ad}\,e)(h))$

$$\mathrm{ad}\,e = \begin{pmatrix} 0 & 0 & -2 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \mathrm{ad}\,f = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 2 \\ -1 & 0 & 0 \end{pmatrix}, \mathrm{ad}\,h = \begin{pmatrix} 2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\mathrm{ad}\,e \cdot \mathrm{ad}\,e = \begin{pmatrix} 0 & -2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \mathrm{ad}\,e \cdot \mathrm{ad}\,f = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix}, \mathrm{ad}\,e \cdot \mathrm{ad}\,h = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\mathrm{ad}\,f \cdot \mathrm{ad}\,f = \begin{pmatrix} 0 & 0 & 0 \\ -2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \mathrm{ad}\,f \cdot \mathrm{ad}\,h = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -2 & 0 & 0 \end{pmatrix}, \mathrm{ad}\,h \cdot \mathrm{ad}\,h = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Thus we have: $\kappa(e,f)=\kappa(f,e)=4,\ \kappa(e,h)=\kappa(h,e)=0,\ \kappa(f,h)=\kappa(h,f)=0$

$$\kappa(f,f)=0,\ \kappa(h,h)=8,\ \kappa(e,e)=0$$

Cartan's Criteria

Theorem (Cartan's First Criterion): The complex Lie algebra, L, is solvable if and only if $\kappa(x,y)=0$ for all $x \in L$ and $y \in L^{(1)}$

Theorem (Cartan's Second Criterion): The complex Lie algebra, L, is semisimple if and only if the Killing form, κ, of L is non-degenerate.

Continuing with the example of $sl(2,\mathbb{C})$ we now assess solvablility:

We will assume the knowledge that $\mathrm{sl}(2,\mathbb{C})^{(1)}=\mathrm{span}\{e,f,h\}$

Since $\mathrm{sl}(2,\mathbb{C})=\mathrm{sl}(2,\mathbb{C})^{(1)}$, we know $e \in \mathrm{sl}(2,\mathbb{C})$ and $f \in \mathrm{sl}(2,\mathbb{C})^{(1)}$.

Therefore by Cartan's First Criterion $\mathrm{sl}(2,\mathbb{C})$ is not solvable because $\kappa(e,f)=4 \neq 0$, thus $\kappa(x,y) \neq 0$ for all $x \in \mathrm{sl}(2,\mathbb{C})$ and $y \in \mathrm{sl}(2,\mathbb{C})'$

Now we turn to semisimplicity:

$\mathrm{sl}(2,\mathbb{C})^{\perp}=\{x \in \mathrm{sl}(2,\mathbb{C}) \mid \kappa(x,y)=0,\ \forall y \in \mathrm{sl}(2,\mathbb{C})\}$

We construct a matrix representation of the Killing form of $\mathrm{sl}(2,\mathbb{C})$ as follows:

$$\kappa(\mathrm{sl}(2,\mathbb{C}))=(\kappa(\mathrm{ad}\,e),\kappa(\mathrm{ad}\,f),\kappa(\mathrm{ad}\,h))=\begin{pmatrix} 0 & 4 & 0 \\ 4 & 0 & 0 \\ 0 & 0 & 8 \end{pmatrix},$$

This is matrix has rank 3, thus its kernel is 0 and so the only linear combination sending these three vectors to 0 is 0 itself, and it is invertible. Thus $\mathrm{sl}(2,\mathbb{C})^{\perp}=0$. We conclude that $\mathrm{sl}(2,\mathbb{C})$ is semisimple by Cartan's Second Critereon.

Categorizing gl(2,$\mathbb{C}$)

We will use the following basis of $\mathrm{gl}(2,\mathbb{C})$:

$$a=\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, b=\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, c=\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, d=\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

Assume $\mathrm{gl}(2,\mathbb{C})^{(1)}=\mathrm{span}\{b,c,a-d\}=\mathrm{sl}(2,\mathbb{C})$, then using similar computations to those depicted with the case of $\mathrm{sl}(2,\mathbb{C})$, we find that

$$\kappa(\mathrm{gl}(2,\mathbb{C}))=\begin{pmatrix} 2 & 0 & 0 & -2 \\ 0 & 0 & 4 & 0 \\ 0 & 4 & 0 & 0 \\ -2 & 0 & 0 & 2 \end{pmatrix}, \text{ and that } \kappa(b,c)=4 \neq 0$$

This only has a rank of 3, therefore there are nonzero elements in the kernel, and thus nonzero elements in $\mathrm{gl}(2,\mathbb{C})^{\perp}$. We conclude that $\mathrm{gl}(2,\mathbb{C})$ is neither semisimple nor solvable.

Applications to Categorizing Lie Algebras

Theorem: Let L be a complex Lie algebra. Then L is semisimple if and only if there are simple ideals $L_1,\ L_2,\ \ldots L_r$ of L such that $L=L_1 \oplus L_2 \oplus \ldots \oplus L_r$

Proposition 1: If L is a semisimple Lie algebra, then $L^{(1)}=L$

Lemma: If L is a semisimple Lie algebra then L is not solvable

Proposition 2: If $L=L_1 \oplus L_2 \oplus \ldots \oplus L_n$ such that each L_i is a simple ideal and I is a simple ideal of L, then $I=L_i$ for some $1 \leq i \leq n$

The above applications are important for a multitude of reasons. We assumed earlier the knowledge that $\mathrm{sl}(2,\mathbb{C})^{(1)}=\mathrm{sl}(2,\mathbb{C})$, and while this is easy to compute by hand, Proposition 1 shows that this property is not a coincidence. We must however still be careful as the converse of Proposition 1 is not always true, as seen through the example of the Lie algebra $L=\mathrm{sl}(n,\mathbb{C}) \ltimes \mathbb{C}^n$. The Lemma stated is derived from Proposition 1, and is a key implication as we conclude that semisimplicity and solvability are mutually exclusive. Thus if we discover a Lie algebra is solvable we know that there is a nontrivial solvable ideal of the same Lie algebra because rad(L)$\neq 0$. The Heisenberg algebra is an example of a solvable Lie Algebra. Direct sum decomposition is interesting as well because the only simple ideals of a semisimple Lie algebra are the ones that occur in the direct sum decomposition. Also we see that even though $\mathrm{gl}(2,\mathbb{C})$ is not semisimple $\mathrm{gl}(2,\mathbb{C})=\mathrm{sl}(2,\mathbb{C}) \oplus \mathrm{sl}(2,\mathbb{C})^{\perp}$, so we can conclude that $\mathrm{sl}(2,\mathbb{C})^{\perp}$ is not a simple ideal, nor does it decompose into simple ideals with respect to $\mathrm{gl}(2,\mathbb{C})$.

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[1] Karin Erdmann and Mark J Wildon. *Introduction to Lie algebras*. Springer Science & Business Media, 2006.